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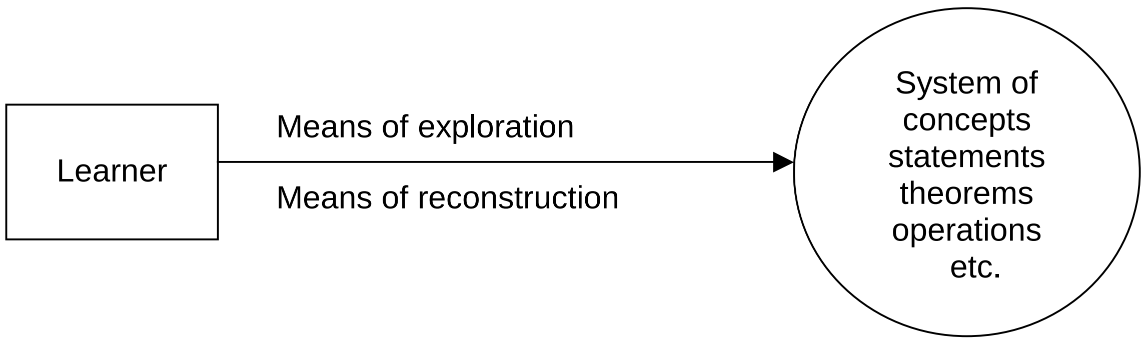
FOR THE DESIGN OF A COMPUTER INTEGRATING GEOMETRY CURRICULUM

The present state of educational software development does have some standardized products, mostly for plane geometry: Dynamic Geometry Systems equipped with international user surfaces. These systems represent a common core curriculum for (middle) secondary geometry independent of specific cultural influences and different styles of interaction (corresponding systems for spatial software are yet less developed). These systems could be considered as a base for a worldwide standard for basic geometric contents and methods, which is available by internet and could be communicatively developed there.

0 DEFICIENCIES OF THE TRADITIONAL GEOMETRIC TOOLS

The traditional tools: compasses, straightedge, set square, rule protractor, paper and pencil and pocket computer as tools of exploration and reconstruction determine largely the usual way of acquiring elementary geometry at school.



The traditional construction, calculation and visualization tools show marked deficiencies as means of exploring and reconstructing elementary geometry by the student.
Such deficiencies of general kind are:
a lack of support of
– epistemic behavior
– individualized learning
– economic working
– visualization
– the formation of flexible and functional thinking
– the development and application of intellectual techniques and heuristic strategies

– modeling real world etc.

By using adequate dynamic graphics systems, these deficiencies can be compensated. The use of such graphics systems however can only complete and not replace the traditional tools, because

– ultimately the importance of handling the traditional analogue tools for the tactile acquisition of fundamental knowledge in geometry cannot be assessed (an experiment where the effects of learning geometry only by means of the computer are explored, is not acceptable for humanitarian reasons)

– working with the traditional tools of construction and of measuring is a cultural technique that is not only very important in the mathematical-historical context;

– the definition of constructive modules with the computer is based essentially on constructive relations as they appear on constructing with the straightedge and the compasses

– there is a risk that only those geometrical subjects are chosen that are especially adequate for being presented and developed with the computer (which is, of cours, also true for the traditional tools in geometry teaching and learning)

– the continuity of specific curricula has to be guaranteed

– the amount of media required in the form of hardware and software causes considerable problems (for example the students require the necessary hardware and software tools to be able to do their homework)

– a worldwide non-verbal communication standard is inherent in these simple tools

– the analogue tools are indispensable for tactile application (do-it-yourself activities etc.).

1 DYNAMIC GEOMETRY SYSTEMS FOR PLANE GEOMETRY

1.1 FACILITIES OF GEOMETRIC COMPUTER SYSTEMS

An interactive graphics system that is considered to be suitable for geometrical construction and calculation in global teaching and learning has to meet some geometrical, organizational and general educational facilities.

1.1.1 GEOMETRIC FACILITIES

Generating geometric configurations

The possibilities of producing geometrical configurations are:

(1) The indirect or passive generation by designing procedures with a graphics programming system or a platform independant script language; the input variables are the determining pieces of construction. We get a formal model of the geometrical construction process like the traditional description of the construction written in a normalized language. The configuration is represented by a program. The procedure called up with actualized parameters produces a corresponding example of the configuration as screen drawing.

(2) The direct or active generation with a graphics system alternately by input of specified commands or options by the user and execution by the system (interactive construction). Two types of graphics systems for interactive construction can be distinguished:

(2a) the command-driven systems

(2b) the menu-controlled systems.

The result of interactive construction is in both cases the target configuration of geometrical construction as a screen drawing.

The systems like (1) can completely describe a configuration using all the control structures for automatic repetition and selection, while by systems of kind (2) only sequential construction algorithms can be executed. The application of the system types (1) and (2a) requires that all the graphical objects to be designed should be named to be able to take reference to them; operation is made with the keyboard and requires the usual command of the semantic and syntax rules of a formal language as in any programming language. Such systems support mainly a graphic handling of constructive elementary geometry aiming at an objective and standardized description of construction. These kinds of representation do not favour selfcorrection by the student but they are very conducive for learning to plan and to imaginate the drawing by anticipation.

The menu-controlled systems (2b) also meet the demands of a "naive user" who uses such tools only occasionally. Menu-driven graphics systems - where direct access to the graphic objects by the user is possible by means of direct manipulation is guaranteed - support a mainly spontaneous approach to elementary geometry which aims at an individual acquisition and exploration. An interactive construction protocol can be issued. There is an option for repetition of construction processes.

The gap between configuration and drawing is bridged by installation of drag mode, which allows easily to represent a configuration by a large variety of isomorphic screen drawings.

Figure generation by direct interaction:

Construction processes traditionally carried out with compasses and ruler

In order to carry out construction processes traditionally realized with compasses and ruler the graphics system must be able to refer to and to generate the following elementary graphical objects:

– points

– straight lines

– circles around a center and through a peripheral point

– point(s) of intersection of straight lines, straight line and circle. and of circles

– points on straight lines and circles.

(If this facility is met, the "method of geometric loci" for construction problems to be solved with compass and ruler can be realized.)

But the mere simulation of compass, ruler and set square construction problems by means of the interactive tool computer in geometry teaching is not sufficient to justify this kind of computer application. Further facilities must offer new possibilities of learning geometry and offer to compensate the deficiencies of the traditional tools.

More basic figures to be generated

In order to be able to represent more general plane shapes the graphics system must be able to generate the following other elementary graphical objects:

– half-lines or rays

– angles (of different kind)

– polygons

– arcs of circles

– conics

and the variety of geometric loci to be individually constructed as referencable objects.

Generation of complex figures

The generation of complex figures derived from basic ones is economically supported by the concept of already implemented or definable macro-construction:

- construction of a perpendicular bisector
- bisection of a straight line segment (i.e. construction of midpoint)
- bisection of an angle (i.e. construction of a line bisecting an angle)

- erection of a perpendicular on a line
- dropping of a perpendicular from a point to a line
- construction of a parallel line
- copying a line segment (replicating a length)
- copying an angle (replicating an angle)

Modification of configurations

Given or constructed figures can be modified regarding their position,orientation, size and shape preserving or changing their incidential structure.

Modification by dragging

Transformations performed by drag-mode are straight line- and circle- invariant. The following relations (if constructively defined within figures) are generally invariant during drag-mode transformations:

– parallelism

– orthogonality

– part-proportionality (i.e. ratio of lengths)

– point symmetry (rotational)

– line symmetry (reflective)

– incidence (in general).

Geometry of drag-mode could be described as a hybrid from aquiformal and affine geometry.

Modification by mapping

Generating figures by application of mappings like congruencies and similiarities as compositions of elementary mappings preserving certain figure proporties.

Modification by redefining

Redefinition of objects serves the change of figure structure for economical construction and investgation, specifying and generalising figures.

Measurement and calculation of configurations

Besides construction of figures a main topic in elementary geometry is measuring and calculating of figures.

Measurement

The following basic measurements, which have to be compatible to drag mode, are indispensible: measurement of

– distances (in relation to a relative length unit as a function of the screen)

– line segments

– arcs

– perimeters of polygons and circles

– angles (of different kind)

– area of polygons and circles

Measurements can be gathered in tables. The input of given measurements is possible. The givens of construction problems can be measurements of angles and line segments. The student uses the measures of line segments or angles for marking off line segments and for laying off angles.

Calculation

Calculations can be derived from measurements by generation of terms.

Calculated data can be gathered in tables.

The values of those tems depending on figure variation by drag-mode.

Macro-calculation

Analogous to the concept of macro construction, macro calculations with quantitative variables can be defined.

Interface for coordinate geometry

There is the possibility of embedding synthetic geometric configurations into a coordinate system and to issue point coordinates and equations of basic graphical objects like straight lines and conics.

Editing of figures

There are editing possibilities for reversable hiding auxiliary objects, for appearance (kind, colour, dimension, position) of objects and for denotion.

Denotion of objects (specifcation):

The graphics system should have a function for naming the graphical objects on the screen and must have the following features:

- freely selectable position of the denomination (automatic positioning of the denomination involves the risk of overwriting)
- denominations as usual in geometry
- optional: free or compulsory denominations.

1.1.2 Organizational facilities

Organizational facilities of an educational graphics system are essential for creating corresponding computerized learning environments and for self organized learning. These environments have to be adapted to the geometrical and instrumental competencies of the students. Modules for creating learning environments can be distributed via the Internet.

- Availability of documented figure files for demonstration and analysis by repetition option
- Availability of documented macro-definitions of constructions and calculations for support of problem solving
- Availability of menu configuration for adapting the system to specific learner groups or to specific problem topics
- Availabilty of menus for non Euclidean geometries.
- Availability of topic orientated interactive worksheets (combination of configurations to be treated according to an explaining text on the screen) with self control facilties
- Availability of information for using the system (self explanation by glossary and online help)
- Availability of different language interfaces to be selected for communication
- Availability of platform independant tutorials with information systems equiped with applets for distant learning.

DYNAMIC GEOMETRY SYSTEMS AS A BASE FOR TUTORIAL SYSTEMS:

The tutorial presentation and preparation of construction, calculation and proof tasks are a prerequisite for self controlled learning in computerized environments.
GEOLOG-WIN/GEOLOG 2000 (http://www.uni-giessen.de/~gcp3/geologde.htm)
is a first in Dynamic Geometry Systems, which integrates tutorial systems on knowledge based way for solving construction, calculation and proof tasks; a correspon-

ding interface serves the tutorial preparation of tasks to be implemented and their sequencing. (Unfortunately, no platform independent version of this system which is programmed in Visual Prolog exists at the moment.)

CINDERELLA offers an interface for preparing and presenting tutorials for all of the compass and ruler based construction tasks at the moment; this tutorial treatment is based on a specific method of mechanical theorem proofing.

1.2 METHODIC IMPACT FOR TEACHING AND LEARNING GEOMETRY

The use of dynamic graphics systems that are on the whole able to meet the already mentioned facilties leads to new methods of learning plane geometry, especially in

– solving geometric construction problems
– insolving geometric calculation problems
– the inductive acquisition of geometric theorems and concept formation
– application and investigation of transformations
– investigation of functional relations of geometric figures
– treatment of elementary functions
– modeling of applications
– simulation of motion in an applicational context
– the aesthetic design from geometric figures.

These new methods are based essentially on the following program features of the systems:

– drag-mode
– macro-concept
– automatic measurement
– generation of terms
– redefinition of objects
– generation of referencable loci
– interface for coordinate geometry
– editing of figures
– administration of results etc.

Using Dynamic Geometry Systems the following general methods are supported

– Variation
– Trial and error
– Modular work
– Generalising
– Synthetic-Analytic transfer.

In the following sections the new methods of an explorative development of plane geometry will be set against the deficiencies of working with the traditional tools.

1.2.1 Solving geomerical construction problems

Deficiencies on solving construction problems with the traditional tools:

– Little support in the heuristic phase of the construction process (e.g. a tentative action in the sense of trial-error-correction is not supported)

– little chance of correcting construction results

– no possibility to change the position or size of the (partial) construction result

– little construction accuracy

– time-consuming processing in case of complex construction processes

– lack of clarity due to the inevitable auxiliar lines

– no possibility of repeating the construction process

– little support in construction due to the lack of construction modules (basic construction processes and user-definable macro-construction: one "great mental step" in a construction process is made up by many small manual steps that divert the students' attention from the aim of construction).

New method:

Solving construction tasks:

(1) construction of a corresponding plan figure

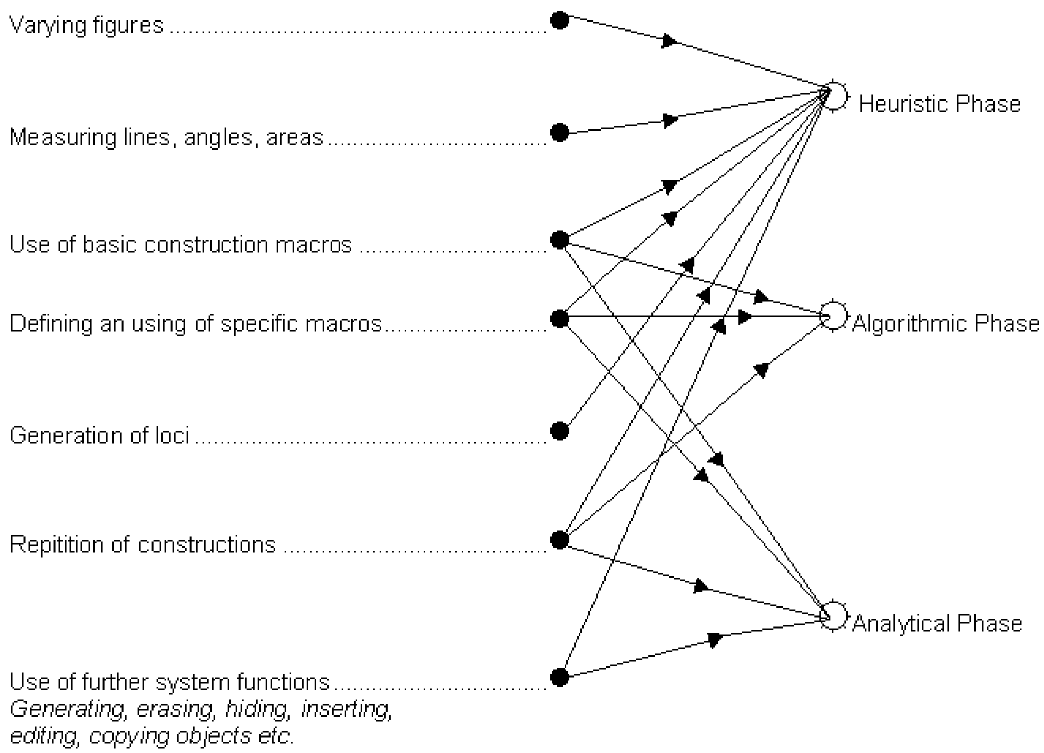
(2) adjust figure in drag mode so, that all of the givens match their predefined quantity

(3) find a constructive solution by heuristic means

(4) simulation of compasses and ruler construction:
the construction steps are analoguely executed as constructing compasses and ruler

(5) definition of the construction macro:
Initial objects are the given graphic and numeric objects; the figure to be constructed from the initial objects consists of the target objects.

If the solution of construction problems is described by a three-phase process: heuristic phase - algorithmic phase - analytical phase all possibilities of the interactive construction tool come into play in these phases, but especially in the heuristic phase of solving construction problems (diagram: an arrow means: ".....supports ...")



1.2.2 Solving geomerical calculation problems

Deficiencies on solving calculation problems concerning drawings of figures with the traditional measurement tools and manual or pocket computer calculation :

– Little support in the heuristic phase of the calculation process (e.g. a tentative action in the sense of trial-error-correction is not supported)

– little chance of correcting and reconstructing calculation results

– no possibility to change the position of the (partial) calculation result afterwards

– little calculation accuracy

– very time-consuming in case of complex calculation processes

– no possibility of repeating the calculation process

– little support in construction due to the lack calculation modules (basic calculation processes and user-definable macro-construction: one "great mental step" in a calculation process is made up by many small manual steps that divert the students' attention from the aim of calculation.

New methods:

– Experimental calculation by using drag mode:

(1) Interactively construct an appropriate geometric figure
(2) Take trial measurements of both given data and target data (also calculated from measurements)
(3) Vary the geometric figure until it matches the given data
(4) Read off the target data.

– Simulation of a calculation process step by step starting with measurements.

– Definition of calculation macros as formulae:

Form a term from measurements or already calculated values of terms concerning a figure drawing, then define a macro calculation which does automatically calculates the value of the formed term from the data given.

1.2.3 Inductive acquisition of theorems and concepts

Deficiencies of the inductive acquisition of theorems by traditional construction of configurations and measurements on configurations:

– time-consuming, often inaccurate construction of a sufficient set of suitable configurations representing the theorem in question,

– only theorems that are based on less complex configurations can be developed,

– time-consuming and incorrect measurements or calculation,

– static configurations that could hitherto in most cases only be made flexible by mental imagination (functional relationships and a dynamic relation of geometrical quantities can be hardly represented).

Analoguely the same applies to the formation of shape concepts with elements from the appropriate geometrized concept ranges.

New method: interactive variation of configurations by changing the position of the constituent objects (so-called basic objects) in drag-mode**.** The initial objects of a construction can be moved freely in drag-mode, all the connected objects follow the movement according to the construction. The transition of the configuration from one state to another is continuous (i.e. in real-time processing) due to individual cursor movements.

The application of the drag-mode offers an opportunity of a real application of the following didactic principles for geometry-teaching:

Realization of the configurative mobility principle: in dragmode the inductive acquisition of theorems or concept formation can be developed by using the following possibilities of continuous variation of geometrical configurations:

Generate from a configuration (as realized theorem or concept) a wide range of many other isomorphic configurations (with continuous transformation. i.e. in real-time processing):

– Generate continuous transformation between special cases of the same configuration.

– Generate from a general case many special cases of a configuration by continuous transformation.

– Generate from a special case more general cases of a configuration by continuous transformation.

– Generate borderline cases of a configuration by continuous transformation.

Realization of the operative principle:

The continuous transformation of geometrical configurations in drag-mode enable a real operative orientation of processes of finding theorems or of forming concepts: Which are the properties of a configuration that remain invariant during continuous individual transformation processes? Theorems from elementary geometry are thus obtained as PROPOSITIONS OF INVARIANCE for continuous transformation of geometrical configurations.

1.2.4 Application and investigation of transformations

Some **deficiencies** in the treatment of transformation using the traditional tools:

– no representation of a transformation as a whole

– time consumimg redrawings for variation of transformation parameters and for combination of transformations

– generating an image of a figure as a whole is not possible.

New methods: Easy generation of image figures; support in the study of tranformational properties and transformational groups by the facilities of dynamic geometry systems as direct manipulation and definition of macros etc.

1.2.5 Investigation of functional relations at geometric figures

Loci are particularly suitable for the examination and illustration of functional relations of geometrical figures.

Deficiencies of the traditional way of generating loci:

– Time-consuming stereotyped repetition of the same construction processes.

– Inaccurate free-hand interpolation.

The construction of loci has been hardly practiced up to now in school-geometry. The wide range of application, the multiple forms and the beauty of plane algebraic curves could therefore not be unveiled to the student.

New method: direct manipulative generation of loci by individual movements of a point on a guide locus, one or more of the points constructively dependent generate the point curve(s) point for point. The operative question which locus (loci) describe(s) the points Y_1, Y_2 ... constructively depending on X if X is moved on a guide loci or freely in the plane can be answered now. The interactive generation curves can be applied to teaching and learning geometry**,** for example

– in the heuristic phase of solving construction problems that can be solved by means of loci method,

– for an experimental verification of assumptions and construction results,

– for investigations on the position and kind of image sets in mappings,

– for explorations of the form curves generated by special points in polygons, especially in the triangle;

– for the construction of algebraic curves (for example for preparing their analytical description)

– for investigation of functions at geometric figures **–** a new link between geometry and real functions:

(1) Constructions of a geometric figure under the condition, that a chozen quantity (as measurement or term from measurements) depends on a quantity to be varied.
(2) Representation of the function relation between independent and dependent quantity as referencable graph of an "empirical" function.
(3) Interpretation of the "empirical" graph (also observation of the graph characteristics at variation of the figure parameters and comparison with other graphs)
(4) Derivation of the function equation, represented by the empirical graph
(5) Control of the derived function equation by testing congruence of empirical and analytical graph
(6) Discussion of the derived function equation (specific values, especially extreme values etc.)

1.2.6 Dynamic treatment of elementary functions

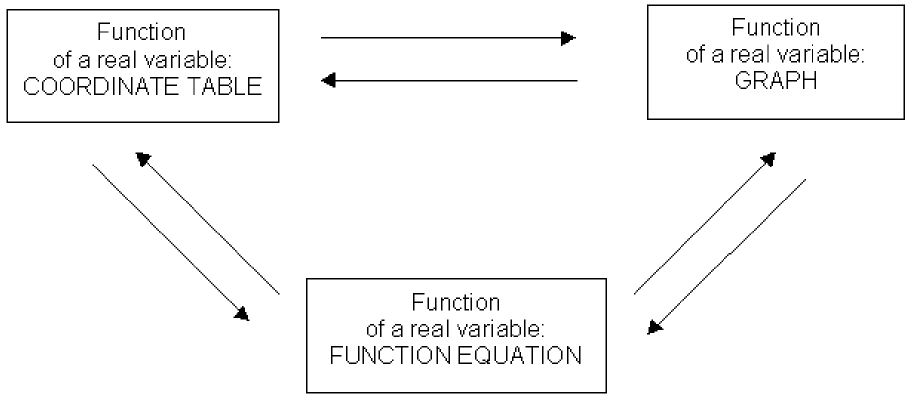
Deficiencies: There are no methods of treating functions dynamically in traditional learning environments already using function plotters.

New methods:

The transition between the three representative modes of a function can be dynamically treated (diagram).

Particularly:

– dynamic production of the graph
– dynamic variation of parameters of the function equation
– dynamic manipulation of the graph
– dynamic variation of coordinates of essential points.



Diagram

1.2.7 Modeling and simulation of applications

Deficiencies: Traditional learning environments do not support the geometric modeling of corresponding engines and the simulation of their action.

New method:

The whole area of flat kinematic applications can be geometrically modelled and simulated now.

1.2.8 Aesthetic geometrical design

The production of beautiful geometrical figures is almost equivalently to the production of symmetric figures . Their individual production can require much stereotyped graphical effort. By the use of corresponding options of dynamic geometry systems this production can be executed time-savingly and playfully.

1.3 PROBLEMS WITH THE USE OF DYNAMIC GEOMETRY SYSTEMS

The following problems result from an uncontrolled use of such systems:

– Loss of (routine) skills of handling the traditional construction tools that are indispensable for entire geometry-learning

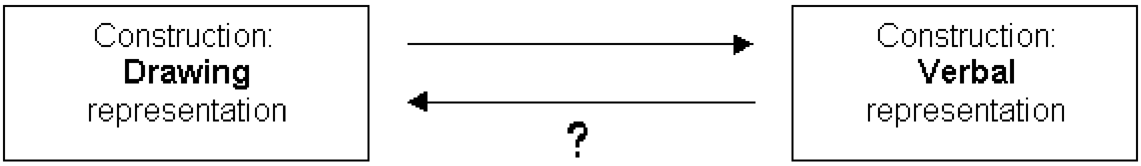
– On solving construction problems, the students tend to be satisfied with an experimental solution obtained in drag-mode. (The following normative requirements remit the students to the simulation of the corresponding compass-ruler construction processes: the construction structure should be invariant with respect to drag-mode and the solution should be completed by defining a macro-construction.)

– Intensification of the problem of motivating proofs of theorems from plane geometry - beyond visual verification (The proporty check facility of CABRI only affirms or denies and does not answer the Why-Question).

– Neglect of the training of verbal representation of geometry.

The following basic restrictions of Dynamic Geometry Systems exist:

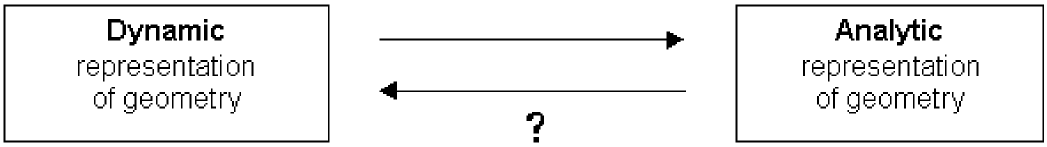
– The manipulative solution of a construction task by the selection of options neglects the aspect of the (verbal) solution planning and the anticipatory imagination of construction steps.



This aspect is created only in GEOLOG and in KONZ: A symbol language is used for the coding of constructions in GEOLOG; constructions can be formed from base constructions in KONZ which consist of verbal phrases with denotions of objects to be adapted to the context of the problem. Unfortunatly commands for the repetition and selection of construction steps don't exist.

– For some geometrical construction problems it is not possible to find a generally valid solution in an interactive way but a correct use of the drag-mode may lead to incorrect construction results (this can be only remedied by coding a suitable generally valid construction algorithm with a user language to be implemented in the system with the usual control structures).

– There is a lack of interfaces distributing an analytic explanation for the synthetic drawing because for the complete translation of the synthetic representation into the corresponding analytic representation and further treatment exist only experimental versions: PLATON (Michael.Schmitz@mathematik.uni.jena.de) with analytic representation in MAPLE and CINDERELLA with analytic representation in MATHEMATICA.



– The system EUKLID DynaGeo (http://www.mechling.de) provides an output of synthetically constructed configurations in terms of the GEOSCRIPT-language –an corresponing extension of Java. That supports the programming of platform independent scripts for synthetic plane geometry beyond sequential algorithms.

– Calculation dependent on drawings don't support the treatment of general geometrical calculation tasks. Solving such tasks can be supported by the use of computer algebra systems. There are two methods for this: the simulation method by means of DERIVE and the formulation method (Ansatz-method) e.g. by means of MATHEMATICA. While the algebraic macro concept of DERIVE has its counterpart in the Macro calculation dependent on drawings, MATHEMATICA supports the expilcit solution planning.

– The problem of the relationship between (real) Euclidean, Hyperbolic and Elliptic geometry is only solved in CINDERELLA from the conceptional point of view.

– Estimating geometrical sizes in principle cannot be integrated into tools because of the necessary tutorial feedback; it requires a particular tutorial software for exercising estimation.

– The transition of the physical tools to the corresponding options isn't sufficiently supported in the computer tools.This problem can be solved by the simulation of the physical tools on the screen. Such simulation exist in Mathlantis Geometrie 1 (http://www.cornelsen.de) und in Klett Mediothek Geometrie I – Mathematik (http://www.klett.de). Mathlantis is a new multimedia geometry learning environment with an integrated dynamic system for construction and measuring and Mediothek “Geometrie“ is a first in computerized information system on school geometry.

–The geometrization of the physical world isn't supported. There isn't any possibility to take off the geometrical structure from a physical represented object. The simulation of this process is particularly well used in the tool Measurement in Motion (http://www.learn.motion.com).

– CabriJava allows the constructive exploring of all the printed figures or figures drawn by hand, which can be imported as html-files; therefore it brigdes an important gap.

1.4 GENERAL EDUCATIONAL ASPECTS AND DYNAMIC GEOMETRY SYSTEMS

Some important educational principles for judging a learning environment which put the acquisition of theoretical concepts independent action in the centre of interest from the point of view of the education of open-minded, independent persons who are capable of communicating, have been taken mainly from the article by O. K. Moore and A. R. Anderson: "Some Principles for the Design of Clarifying Educational Environments" (in: A. Goslin (Ed.): Handbook of Sozialization Theory and Research. Chicago 1969, pp. 571-613).

A graphics system for geometrical drawing in the classroom should meet the following educational demands:

– **the principle of different roles**

– **the autotelic principle**

– **the principle of productivity**

– the principle of personalization.

The following pages present a short description of the individual principles, and outline and interpret the demands that have to be met by the learning environment of an interactive graphics system for geometrical construction in the classroom (in comparison with the learning environment offered by the traditional construction tools). The important part played by the teacher who has to organize the learning processes in the corresponding learning environment must be pointed out here.

The principle of different roles:

An environment is more favorable for learning than another if it permits the user to play a greater number of roles with regard to the matters that have to be learnt.

The following (ideal-types of) roles can be distinguished:

– the role of the active person

– the role of the reacting person

– the role of the consultant

– the changed role

– the role of the partner.

The system should offer a better support of the student in her/his role as active designer. In the role of the reacting person, the student could find more possibilities of learning by imitation if the teacher (or another student) demonstrates construction processes with the system or handles construction results in a flexible way. The dynamic repetition of the construction processes and the construction description in the standardized form are preconditions for an easier judgment of the student's own construction results or of the construction results of other students. The intention of such systems is to encourage students also to play the teacher's part, for example in demonstrating construction processes or in finding construction problems. The system wants to give impulses for a communication on the teaching contents on equal terms.

The autotelic principle:

An environment is more favorable for the learning process than another if it encourages the student more than to indulge in activities that have their end in themselves.

The students who are working with the system should be encouraged to start activities of playful construction and manipulation of geometrical shapes up to serious and consistent construction for example on solving a difficult construction problem or on (subjective) discovery of a geometrical proposition. (From the phenomological point of view this may be recognized in the consistent motivation to work with the system. e.g. if given limits of the contents or of the time available are ignored). A long-term improvement of the general attitude towards geometry should become apparent.

The principle of productiveness:

An environment promotes the learning process more if it offers greater possibilities of expansion, transfer and application of the learning contents to be acquired.

The user's competence obtained by the use of the graphics system for geometrical construction should permit a transfer and application to the use of other, more comprehensive graphics programs, e.g. CAD-programs, programs for advertising art, etc., that are more and more frequently used in the professional world. - The system should favor the application of heuristic strategies for solving geometrical problems.

An improved (intra-mathematical) transfer of heuristic strategy application should take place.

The personalization principle:

An environment favors the learning process more if it responds more readily to the students activities and encourages the student's reflection of their learning process.

The graphics system should allow an improved self-controlled learning-process (which is among other things achieved by the flexible "student-system" interface and the system aids). The possibility of correcting errors without much effort should help the student to understand that the method of trial and error and the errors committed are constructive elements of her/his learning process. - The system's protocol and diagnostic functions allow a deeper reflection of the learning process by the students.

On the whole, the constructive perception process which roughly takes place in the three phases: "epistemic conflict – self-reflection – self-correction or regulation" should be encouraged by the use of the system.

2 DYNAMIC COMPUTER SYSTEMS FOR SPATIAL GEOMETRY

2.1 Introduction

Experience teaches us that representing space geometric facts on paper, at the blackboard or in the form of material models is laborious and often unsuccessful even if certain techniques of representation or design have been practised.

The flat representation of a spatial figure doesn't have any spatial depth; it is statically and hardly correction-capable; it cannot be "manipulated"; it can only insufficiently adapted to a learning and teaching process or a process of exploration etc. These weaknesses of conventional media favour the lesson time for plane geometry against the lesson time for spatial geometry (in the teaching curriculums we mostly find only these topics that can be well created with the traditional media in the lesson)! Therefore the computer representation for spatial geometry is added to paper and pencil and print media representations and material representations like physical models of solids. However, this causes new interface problems.

The diagram 1 illustrates the relation of the three modes of representation with their corresponding interfaces S_i , S_i', i=1,2,3.

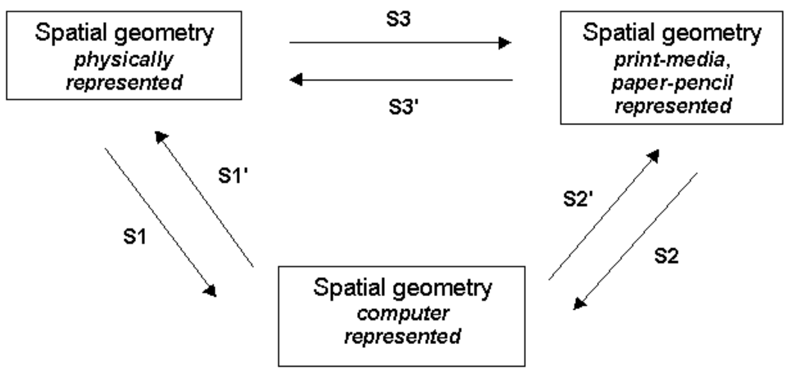


Diagram 1

The significant change in the treatment of spatial geometric problems:

In general, we can solve spatial geometric problems in the conventional learning environment only by means of the solution of corresponding problems of plane geometry developed by methods of descriptive geometry (Diagram 2).

With the computer use we have the possibility to produce and to represent spatial geometric configurations on the screen with virtual spatial depth and to directly manipulate these configurations in drag mode (Diagram 3). This simplifies solving spatial geometric problems considerably and avoids the "traditional detour".

Remarks about the interfaces:

S_1/S_2: If the solid represented in a drawing or provided as a material model is available but not already as a digital model in the computer tool, how can it be implemented? If necessary we must express and calculate the corners of the solid in question in a three-dimensional coordinate system (a laser beam scanning of material objects doesn't seem to be a practicable digitalization method in the school geometry).

S_1^I: Spatial geometrical screen representations can be printed and documented in simple ways.

S_2^I: How do we get from a spatial object on the screen only accessible to the visual perception of a material object which can also be refered to haptically – but not considering the perception possibilities of the Cyber-Space here? The practicable solution consists in the generation of solid nets on the screen, being able to be printed and then folding them up as surface models. However, this solution of this interface problem does not work for the sphere and its parts etc.
The methods of the Solid-Imaging are still not available for school.

"Spatial geometry - conservatively"

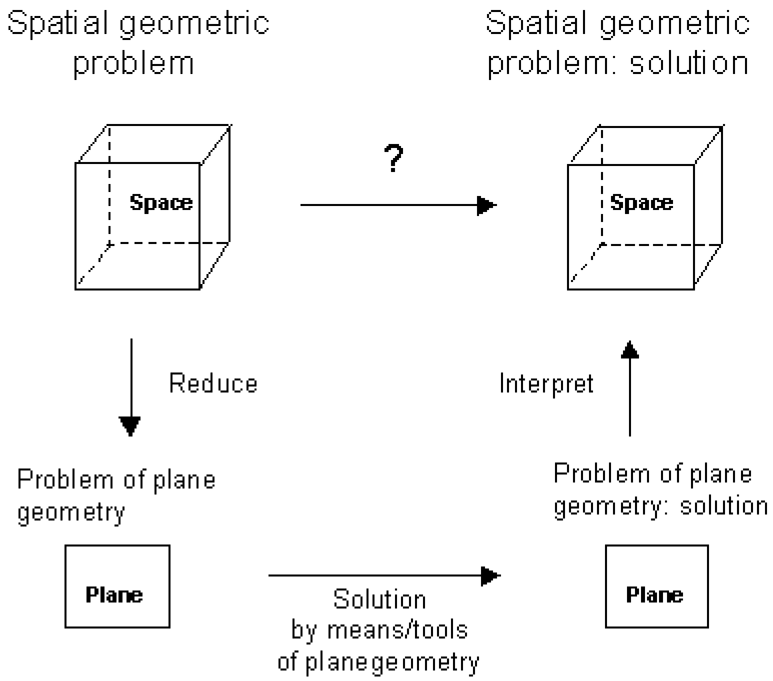


Diagram 2

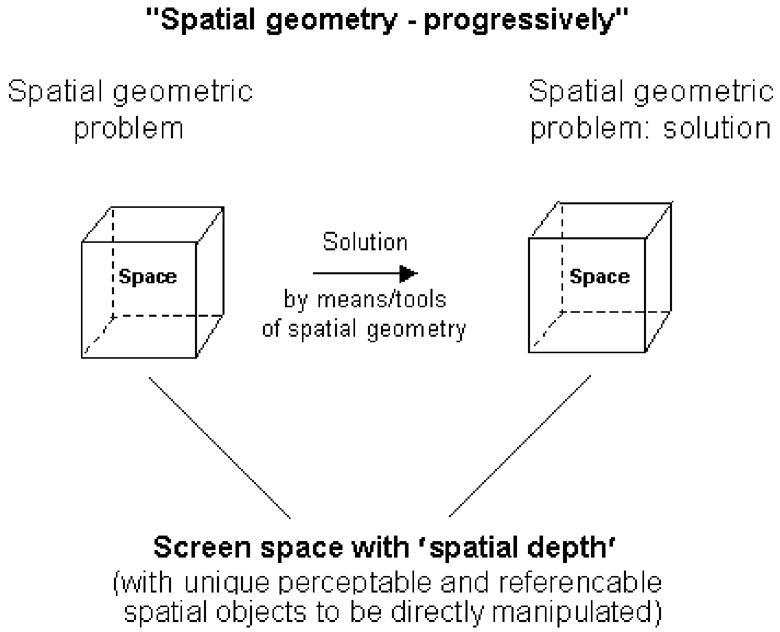


Diagram 3

2.2 Prototypic computer graphical tools for spatial geometry in school

Two software development lines become apparent. On the one hand, tools are developed that are essentially restricted to the representation and processing of solids, and on the other hand tools, that allow essentially spatial construction like dynamic geometry systems plane geometry.

Until now the latter tools have been completed only for Macintosh (3D-Geometer, Klemenz 1994/99). -with the defect of the clear perceivability of spatial objects in the depth (e.g. the relational position of the sphere and a straight line: all cases must be perceptible visually!). MiniGeometer, derived from 3D-Geometer, is a Java-applet for interactive construction in spatial geometry (http://geosoft.ch); it doesn't contain any improvement in the mentioned problem of perception, and does have a quite complex user interface which is limitation of its use for middle and early secondary education. A hopeful development, the Macintosh program Cabri-Géomètre 3D, still isn't completed.

So, which essential demands must now fulfil a computer graphical tool to be used for learning and teaching spatial geometry in secondary education that mostly deals with geometric solids?

- As a visualization tool it allows among other things the possibility to look at the standard solids of school geometry and in addition the solids derived from these solids as if one would have these solids as edge-, surface- or full solid-model in the "hand".
- (This is managed by Virtual Sphere Device, this means that a referencable virtual sphere circumscribed around any solid can be arbitrarily moved with the mouse.)
- It makes possible the transition from only visually perceptible solids on the screen to the haptic perception of.them. (This happens by printing of solid nets and then folding them up.)
- As a measuring tool it allows the study of metric properties of a solid in various ways and allows the investigation of the true form of objects which are on or in a solid.
- As a construction tool it allows a flexible creation of solids by dissection, composition, deformation etc.

  In addition it must be possible to draw figures into and onto the solids to make them carriers for further geometric information.

Therefore the Windows programme KOERPERGEOMETRIE (Bauer, Schumann et al. 1999), which essentially fulfils the mentioned demands can provide:

- the demonstration of solid geometric facts
- the support of the knowledge of spatial shapes, the constructive representation, the calculation and the production of geometric solids
- the development and the training of spatial ability (here: ability to imagine spatial objects and relations between spatial objects)
- the experimental finding of knowledge (discovery of geometric statements, production of new solids etc.)
- the reinforcements of working creatively by spatial geometric exploration (e.g. finding the solution of open problems).

There is one more development currently available in German area: The tool SCHNITTE supports only the visualization, the representation, the sectioning step by step and the automatic folding up of convex polyhedra; it has simple user surface, but it is of less direct manipulative quality. –The three-dimensional module of the tool SHAPE UP! is only restricted to the NCTM standards (Rappoport, E. et al (Eds.) (1995): Shape Up! (Software). Pleasantville, NY: Sunburst). In the following we refer to the tool KOERPERGEOMETRIE (an English user surface is in work), which contains a more rich environment for spatial geometry.

2.2 An Exemplary Study: the square pyramid

Unlike the handbook in which the options of KOERPERGEOMETRIE are explained in detail, the didactically oriented treatment of some solids shall be focused here. For our tour, examples of the square pyramid and then the cone and the cylinder have been selected first. (The content of this tour can be used also as a topic for a computer supported project in learning spatial geometry.) Remark: The print media documentation of the tour is insufficient to represent the dynamic work of the computer graphics tool such as KOERPERGEOMETRIE.

2.2.1 Visualization of square pyramids

From the tool bar option**“basic solids“** we select a square pyramid among the basic solids: cube, cuboid, parallelepiped, triangular prism, cylinder, triangular pyramid, cone, sphere, square frustum, frustum of the cone, half cylinder, hemisphere (Figure2.1).

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Figure 2.1 The pyramid monochrome or multi-coloured can be turned with the mouse to look at all sides (Figure 2.2.1-2.2.3).

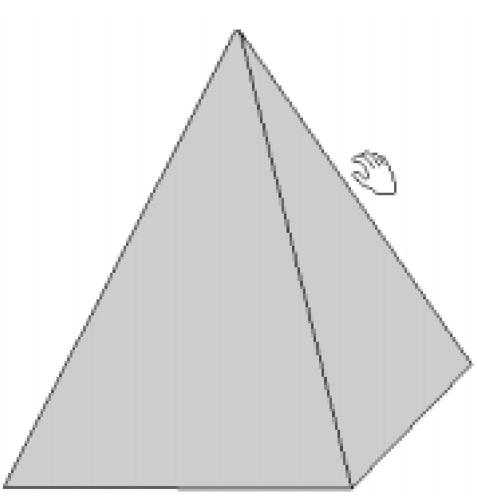


Figure 2.2.1

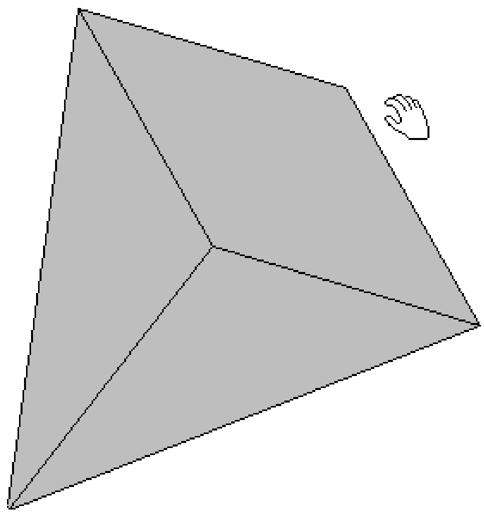


Figure 2.2.2

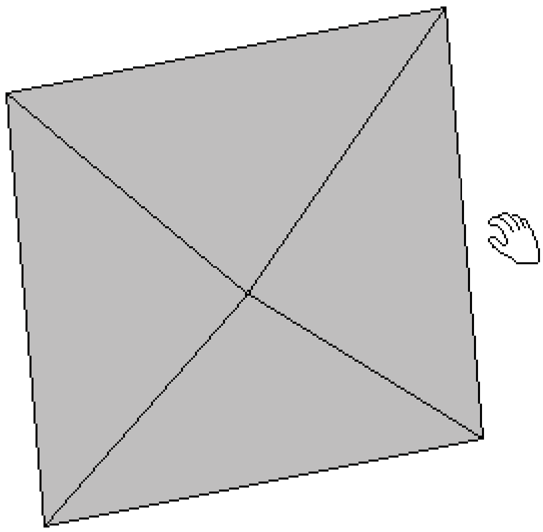
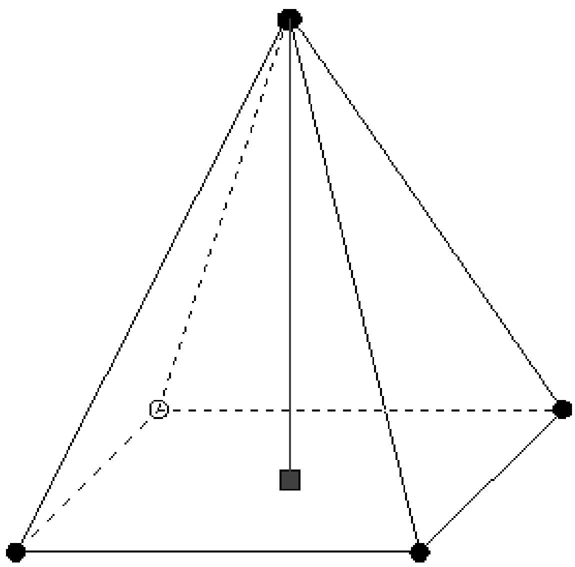


Figure 2.2.3

After drawing the height into the surface model (tool tool bar **„Production of solid points and distances "**) we can fix the height as axis of rotation (tool tool bar **„Move of solids“**) and turn the pyramid around this axis (Figure 2.3.1/2).



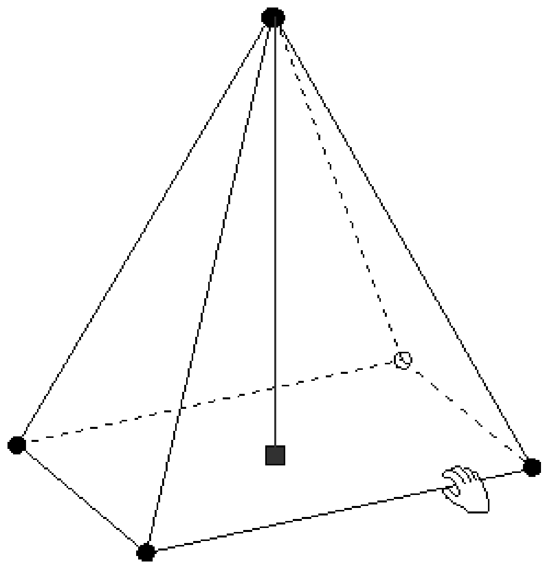
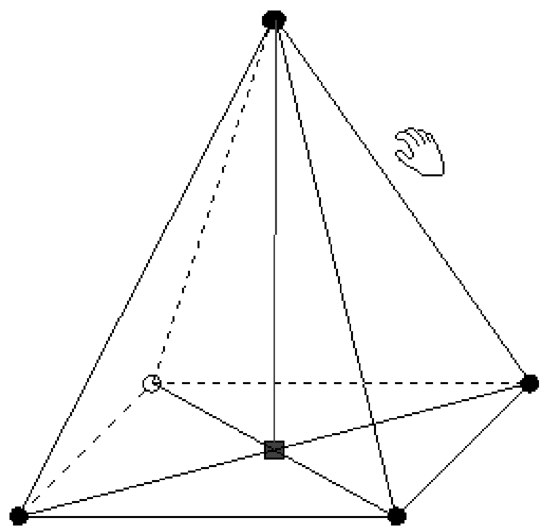


Abb 2.3.1 **Figure 2.3.2**

Here and in the following we use the usual parallel projective image representation of a solid on which the covered edges are dotted. –The position of the height with the marked diagonals of the base is watched when turning around the predefined rotational centre (Figure 2.4.1/2).



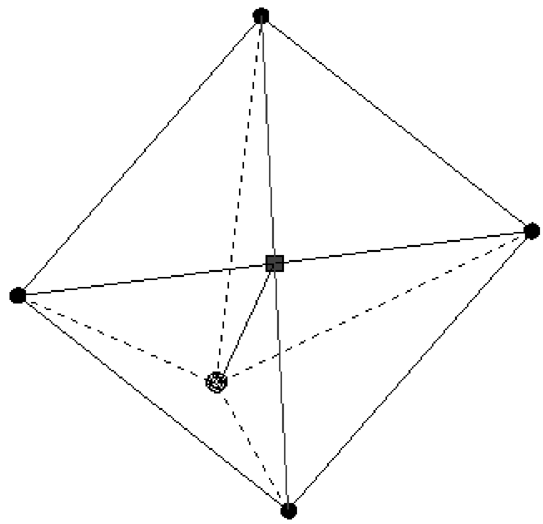


Figure 2.4.1 **Figure 2.4.2**

In the tool bar **„Representation of solids“** we can select the spatial corner and see the vertical projection of the pyramid on the three planes (Figure 2.5.1 with projection lines).

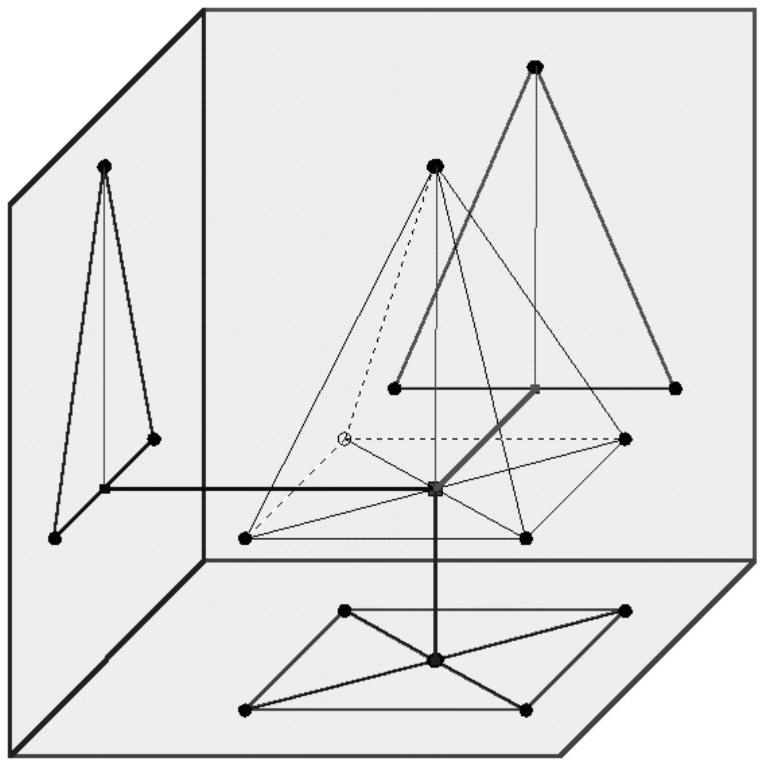


Figure 2.5.1 By means of animation the spatial corner is made flat (Figure 2.5.2),

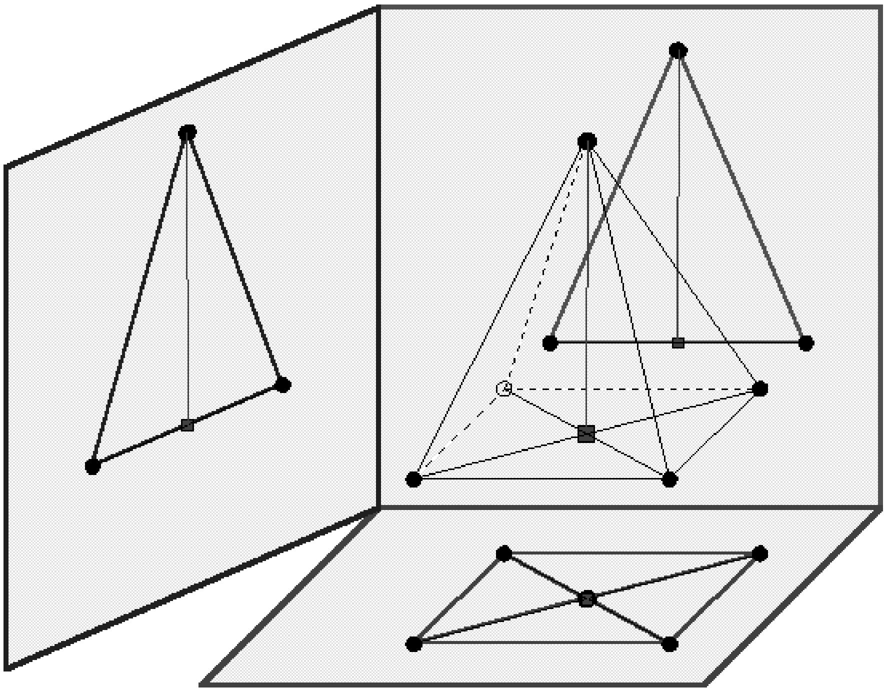


Figure 2.5.2

We can put the square pyramid besides its three plane projections (Figure 2.5.3).

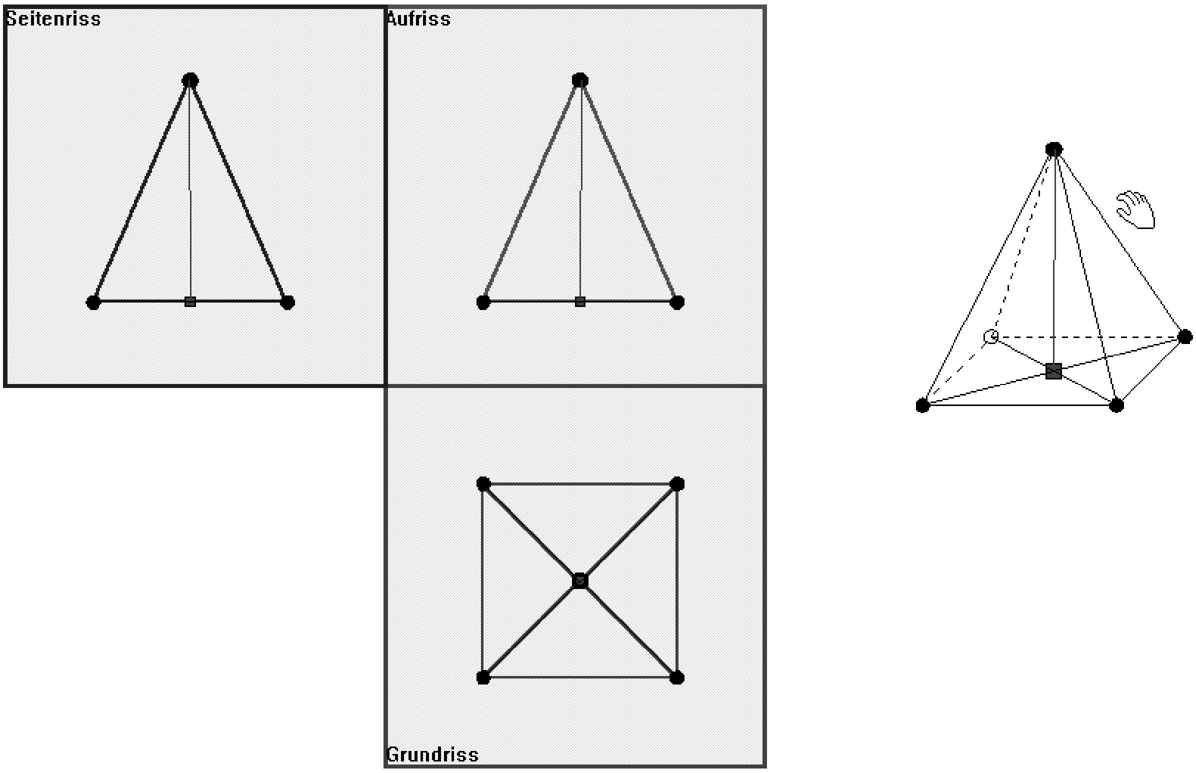


Figure 2.5.3

and watch the consequence of its rotation (Figure 2.5.4).

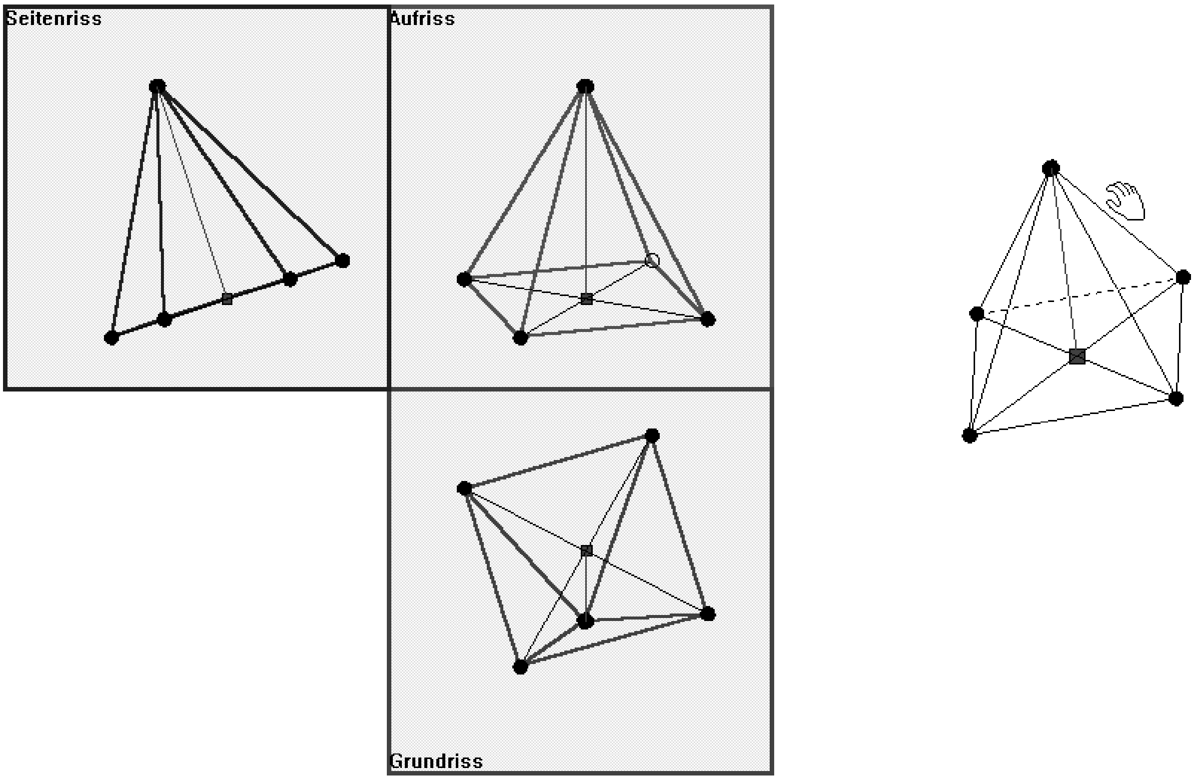


Figure 2.5.4

Of course n-sided pyramids could be flexibly dimensioned (Figure 2.6.1 "Solid description..." under menu point "Generation").

Figure 2.6.1

An equal edged square pyramid shows the figure 2.6.2.

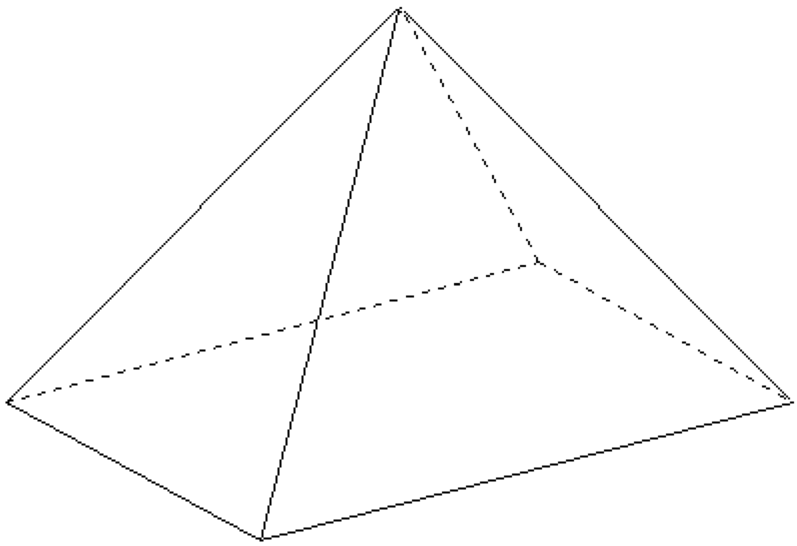


Figure 2.6.2

A corresponding dimensioning is also possible for prisms, pyramid frustum, double pyramids and the various parts of the sphere.

In th tool bar **„Measurement / Generation of solid nets“** solids could be automatically (Figure 2.7.1) or interactively, and thus individually, unfolded to make a net (Figure 2.7.2).

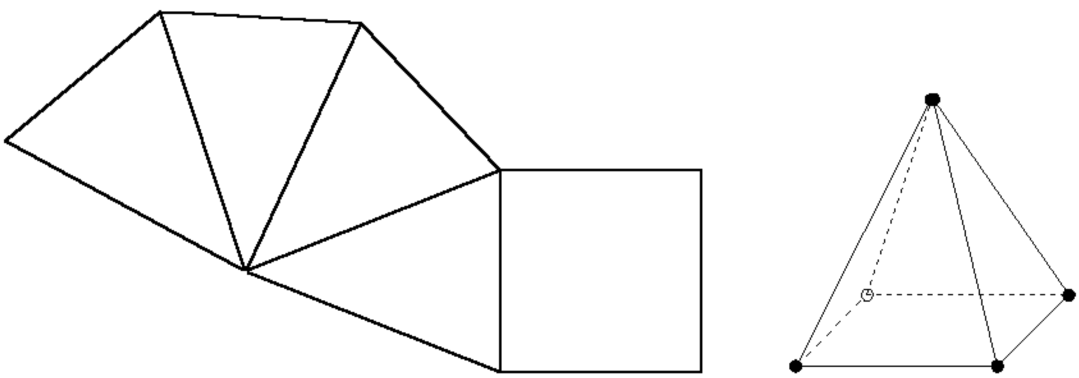


Figure 2.7.1

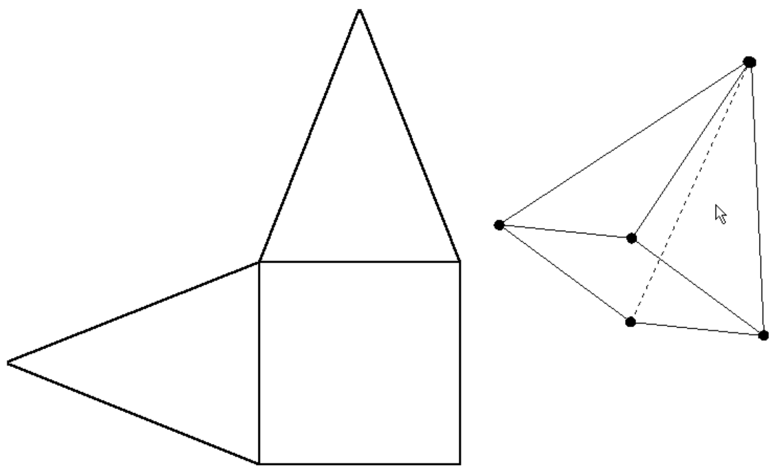


Figure 2.7.2

One can decide when unfolding which side shall be fastened to which side of the net, e.g. the eight different nets of a square pyramid can be discovered in this way. –We have the possibility now, with printout of a net and cutting it out, folding it up and fix it e. g. by means of adhesive tape, to tactilly experience the surface model represented on the screen (e.g. look into its inside through an open "side window").

2.2.2 From square to the "quadrilateral" pyramid

We are able to generate new square pyramids e.g. by dragging its peak. Therefore we start from a square pyramid with its symmetric net (Figure 2.8.1) and generate a square pyramid with edges of equal length (Figure 2.8.2).

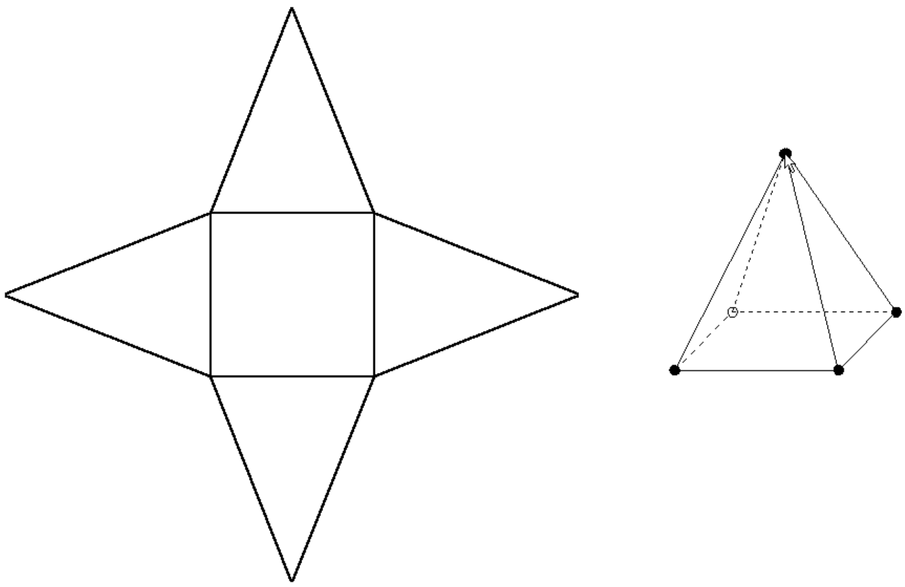


Figure 2.8.1

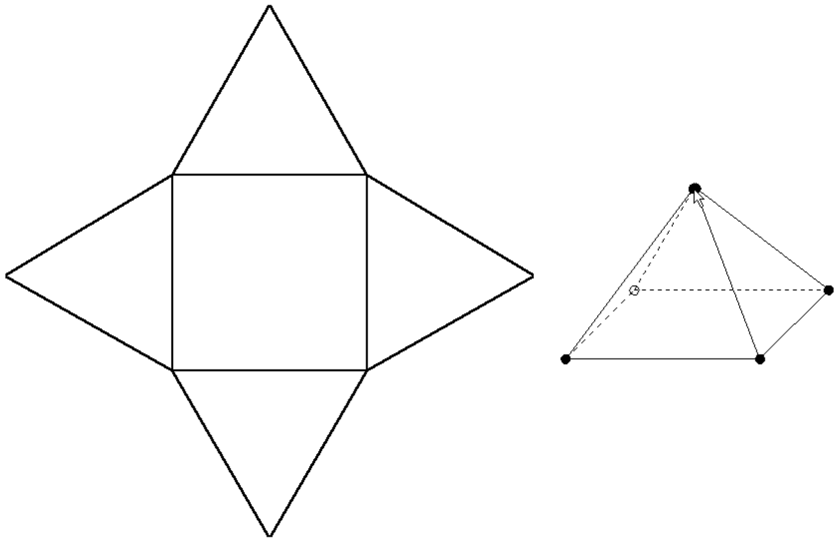


Figure 2.8.2

Or we drag to produce a pyramid for which the foot point of the height doesn't coinci-de with the centre of the base any more (Figure 2.8.3, with a symmetrical net),

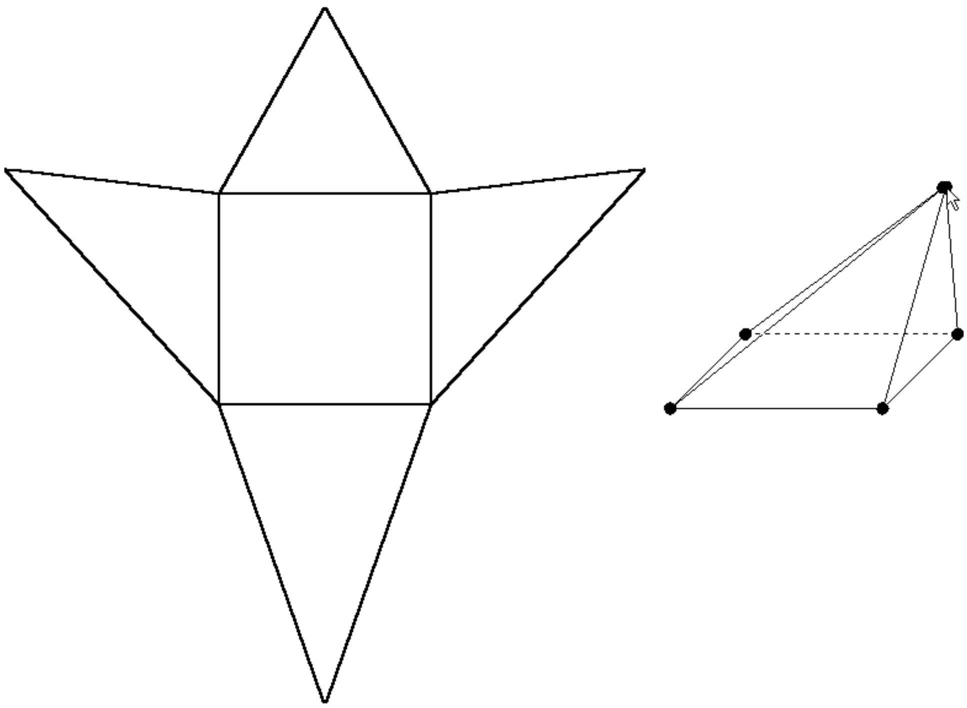


Figure 2.8.3

or to produce a pyramid for which the foot point of the height lies outside the base (Figure 2.8.4 with an asymmetrical net).

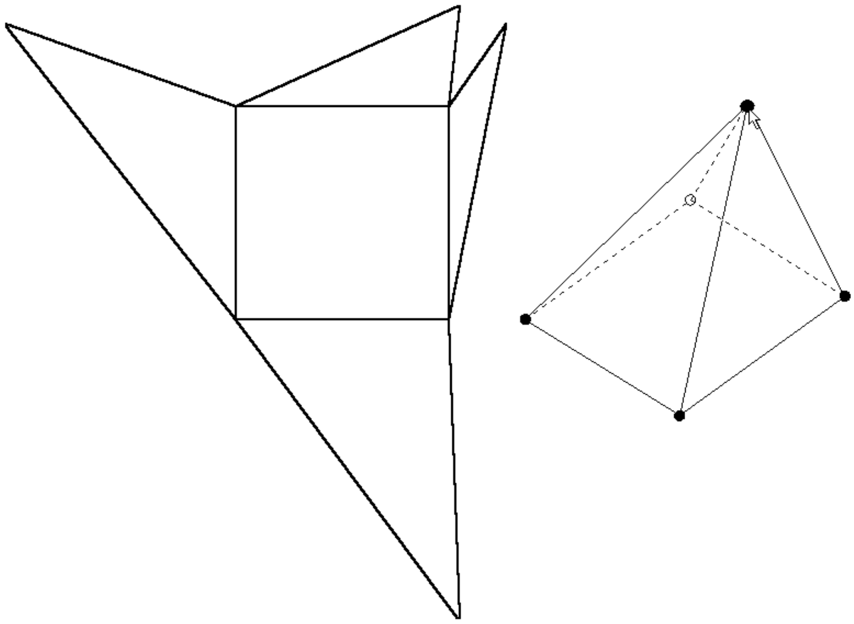


Figure 2.8.4

Now we also drag the square base out of shape to produce any convex quadrangle (Figure 2.8.5). The printout of the net can be used for the construction of the foot point of the height: The perpendicular lines from the tops of the triangular sides on the corresponding base edges or their prolongations intersect each other in this point.

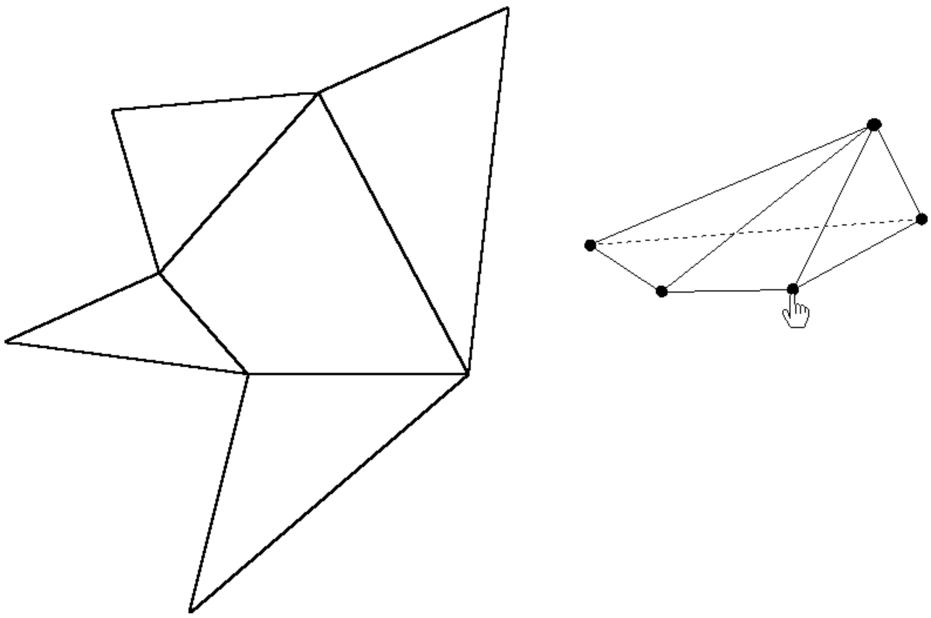


Figure 2.8.5

With the options of the tool bar „Move solids" solids can be reflected also at points and in planes. If one reflects a non axis symmetrical square pyramid (e.g. through a vertical plane), then its reflection has a mirrored net (Figure 2.9.1/2).

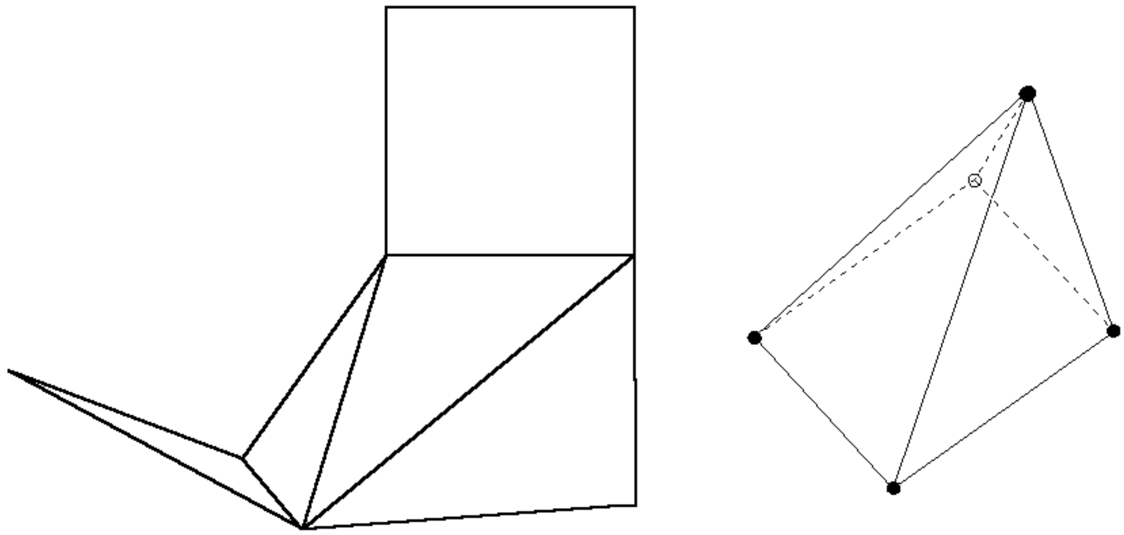


Figure 2.9.1

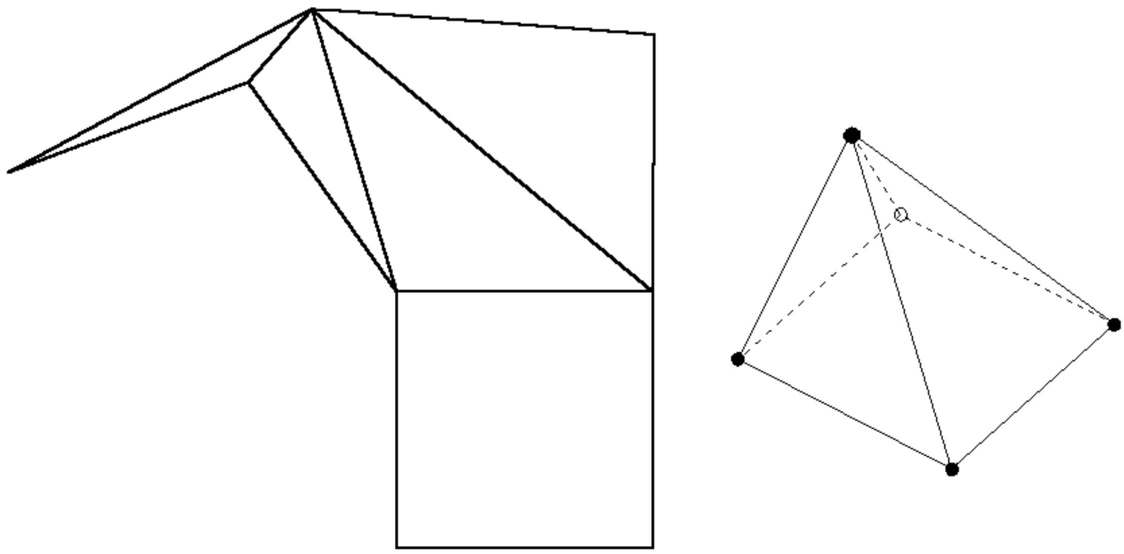


Figure 2.9.2

Remark: With the options "Reflect at a plane" and „Rotate around an axis " one can study the self mappings of a solid and their combinations. For this purpose the vertices of the solid are colourable.

2.2.3 Measuring pyramids

Any selected solid can be measured: One moves the cursor on the solid then a report is provided of the area of the faces, the lengths of the edges, the volume and the size of the total surface area. In addition; for the standard solids of course still more measures are given (Figure 2.10).

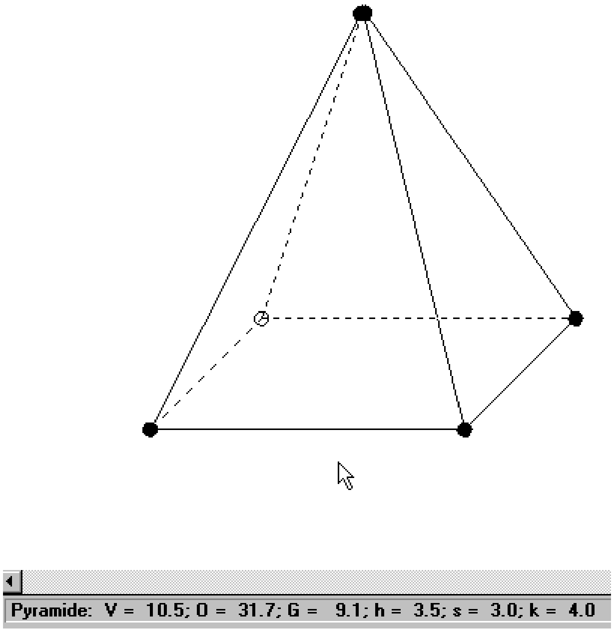
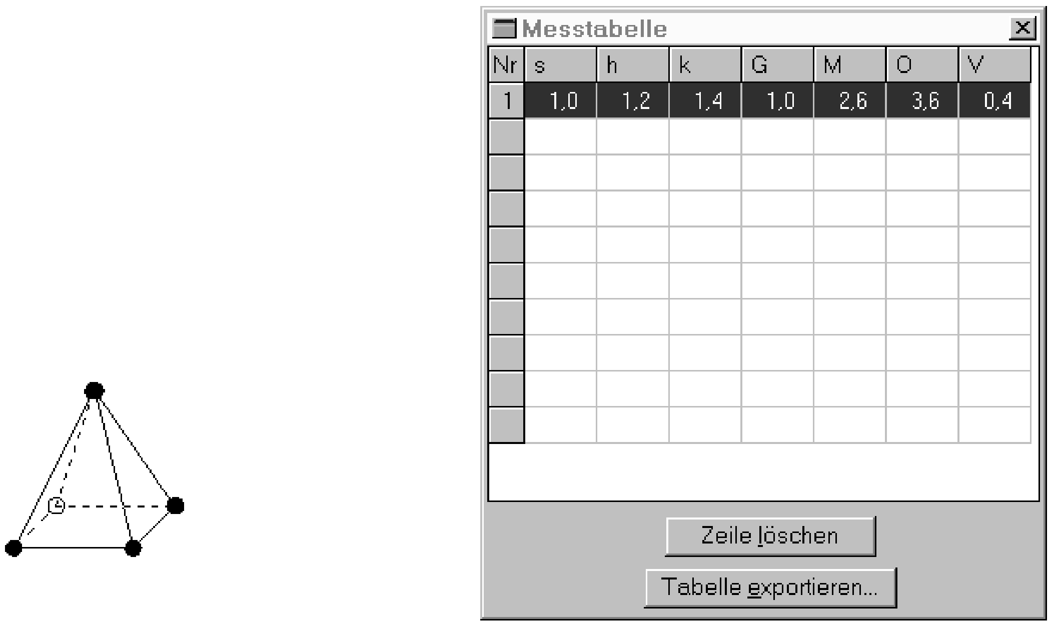


Figure 2.10

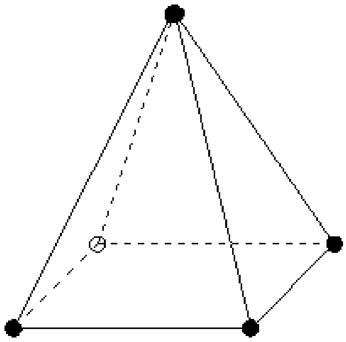
The current measures of a solid can be displayed also in a table. For this one establishes a table with a corresponding header in the tool "Measurement / Generation of nets“ and all the actual results are brought together (Figure 2.11.1).



Nr	s	h	k	G	M	O	V
1	1,0	1,2	1,4	1,0	2,6	3,6	0,4

Figure 2.11.1

We use this facility to investigate the rule of variance for surface and volume of the square pyramid under similiarity transformation.



Messtabelle

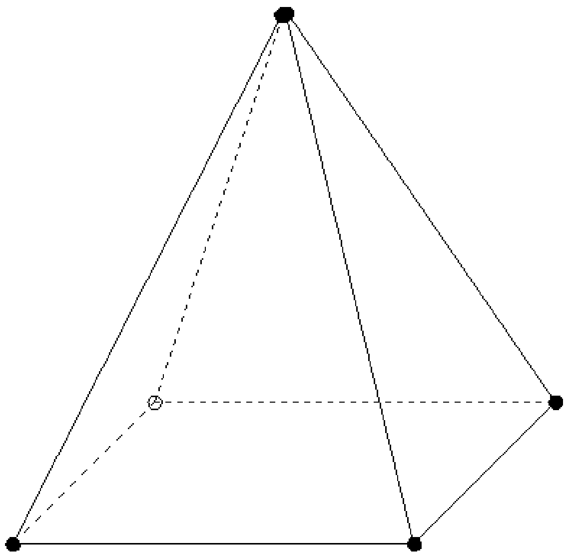
Nr	s	h	k	G	M	O	V
1	1,0	1,2	1,4	1,1	2,7	3,8	0,4
2	2,0	2,3	2,7	4,1	10,3	14,4	3,2

Zeile löschen

Tabelle exportieren...

Figure 2.11.2

We can, also use of a online-calculator: We extend the pyramid until the base edge s is of twice its length, the other lengths also double themself, the area measures grow on fourfold and the volume increases eightfold (Figure 2.11.2/3).



Messtabelle

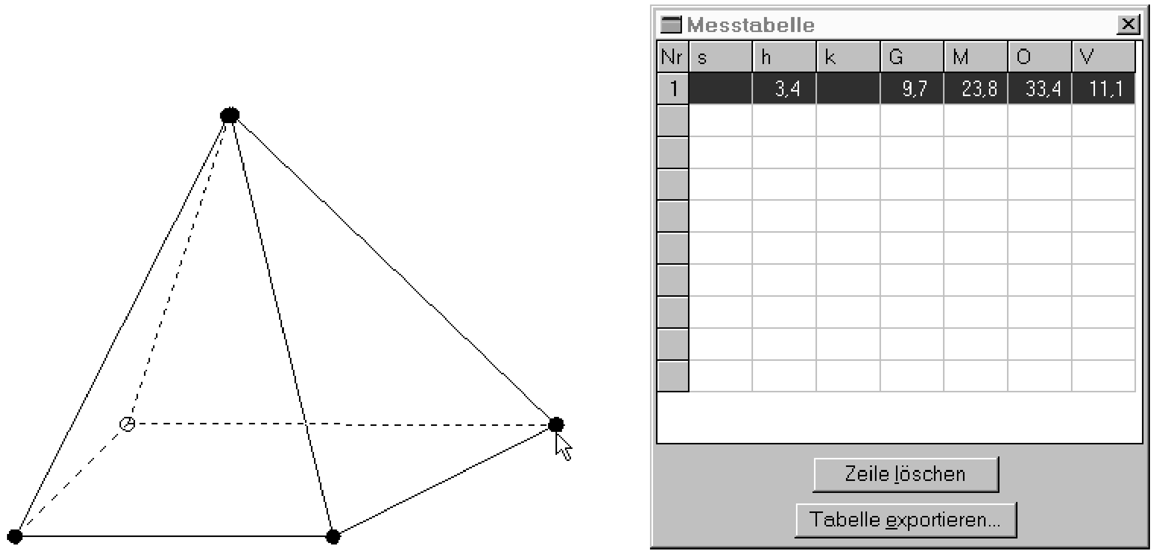
Nr	s	h	k	G	M	O	V
1	1,0	1,2	1,4	1,1	2,7	3,8	0,4
2	2,0	2,3	2,7	4,1	10,3	14,4	3,2
3	4,0	4,6	5,4	16,0	39,9	56,0	24,5

Zeile löschen

Tabelle exportieren...

Figure 2.11.3

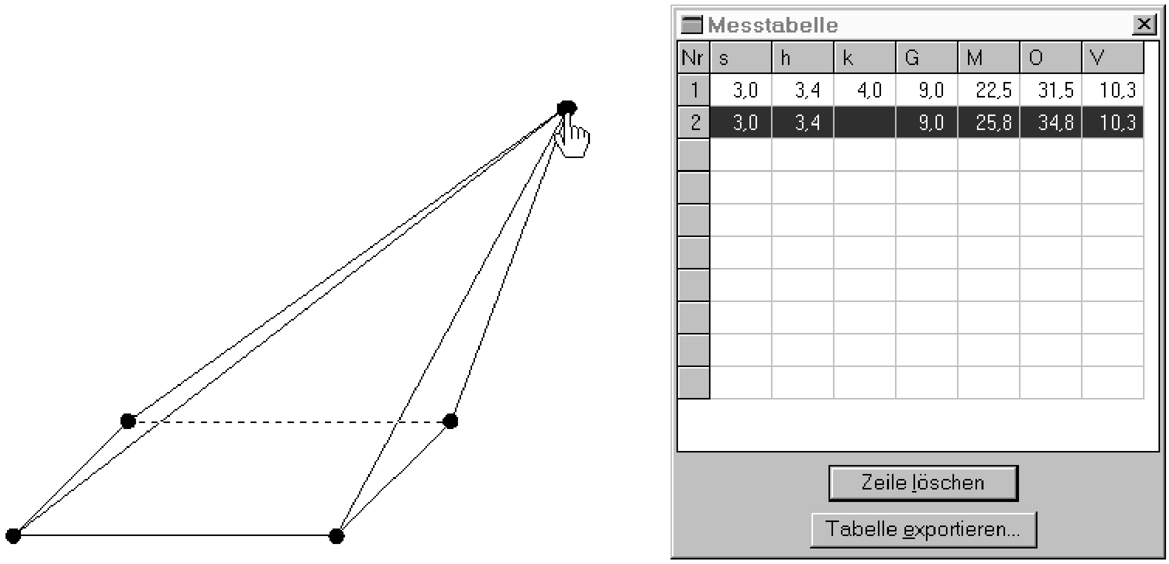
If we drag the square base of the pyramid to any quadrangle (e.g. to a trapezium) then the displayed results for base and lateral edges disappear inevitably (Figure 2.11.4).



Nr	s	h	k	G	M	O	V
1		3,4		9,7	23,8	33,4	11,1

Figure 2.11.4

If we displace the position of the top but maintaining a constant height (spatial shearing), then we observe the invariance of the volume (Figure 2.11.5).



Nr	s	h	k	G	M	O	V
1	3,0	3,4	4,0	9,0	22,5	31,5	10,3
2	3,0	3,4		9,0	25,8	34,8	10,3

Figure 2.11.5

In the tool bar **“Objects in true shape“** objects drawn in or on a solid can be seperately displayed in their true form and size, both graphically and numerically; in addition, such objects can be moved parallelly or rotated around axes.

For example the true size of the height of a triangular side together with its measurement can be shown (Figure 2.12).

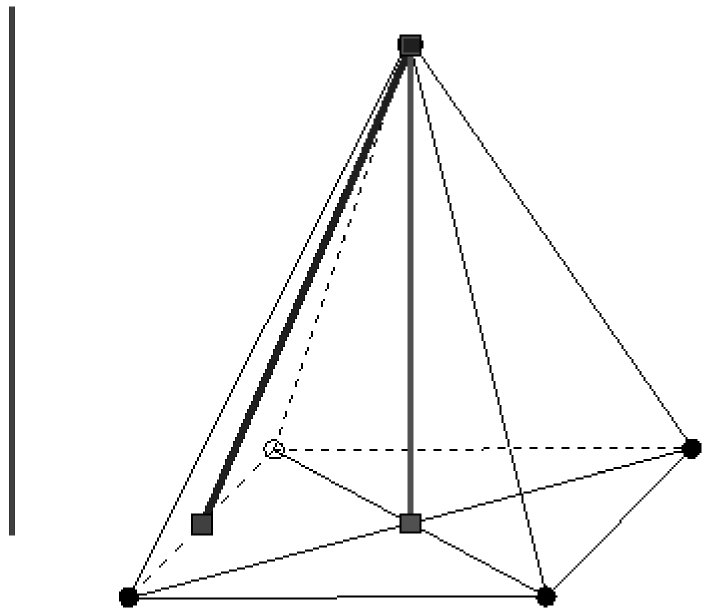


Figure 2.12

The same is possible for the size and area of a triangular side (Figure 2.13) and for

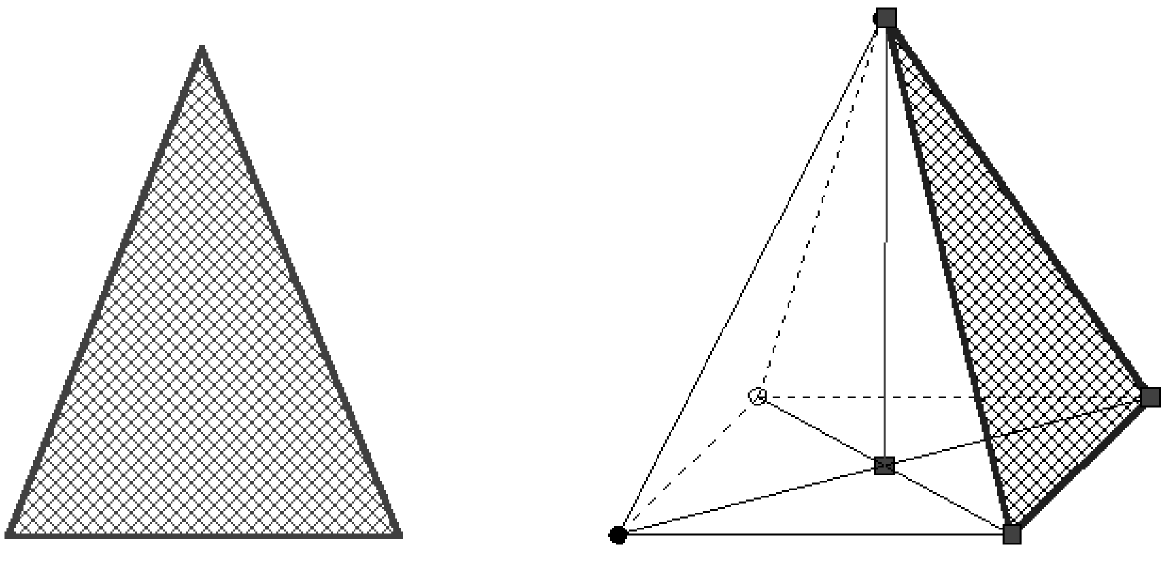


Figure 2.13

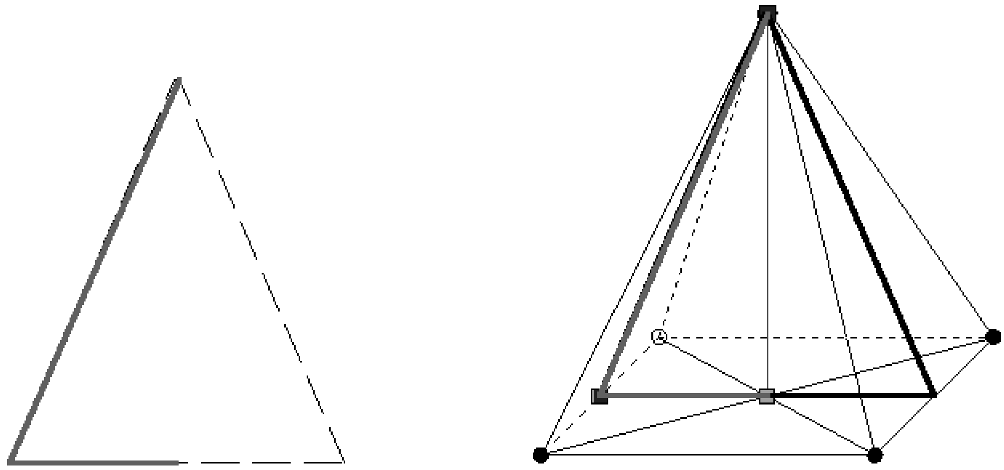


Figure 2.14

the angle between side altitude and base (Figure 2.14), and the angle between lateral edge and a base diagonal (Figure 2.15):

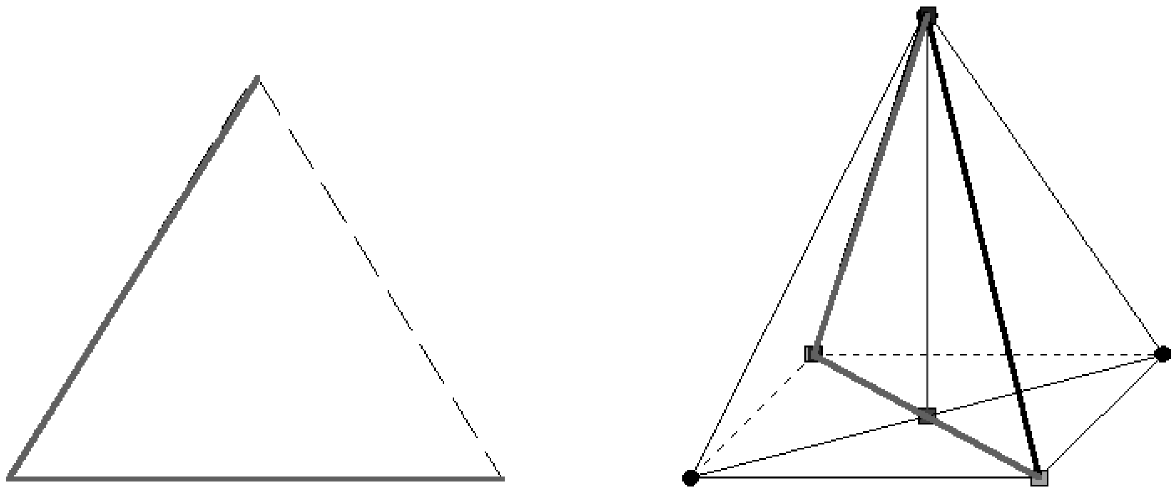


Figure 2.15

We create a pentagonal section in the pyramid defined by three lateral edge points and two base edge points and see the one axis symmetrical section represented in true form (Figure 2.16.1).

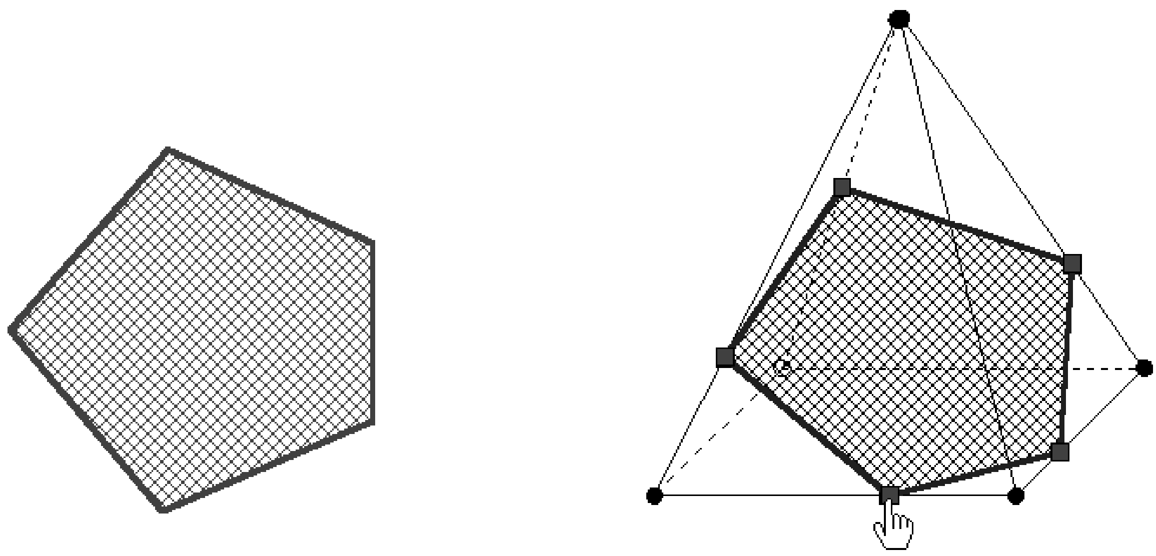


Figure 2.16.1

On request the solid rotates automatically so that this area assumes its true form inside this solid (Figure 2.16.2); one also can try this by manual controlled rotation, for which the angle of deviation is shown.

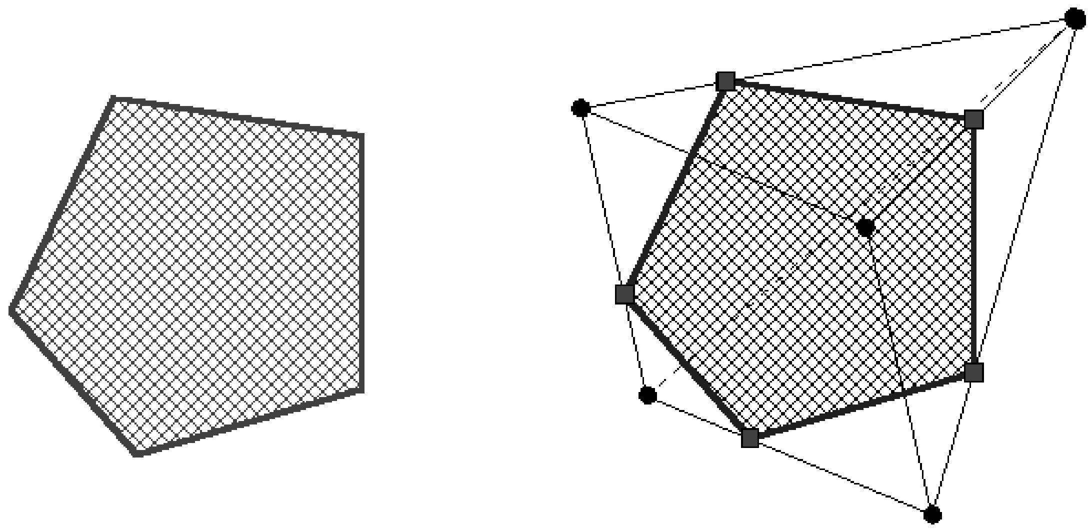


Figure 2.16.2

We vary the form of this pentagonal area by parallel moving (Figure 2.16.3, producing a kite – for example).

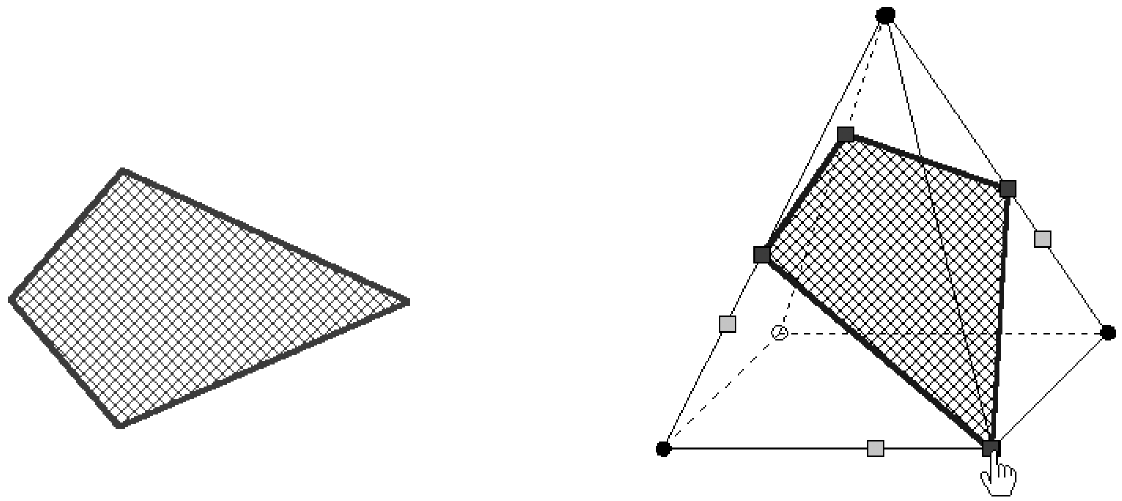


Figure 2.16.3

How do the sides of the polygonal section appear in a net of the corresponding solids and how do they change? The illustrations show this for the parallel moving in examples in figures 2.17.1 and 2.17.2.

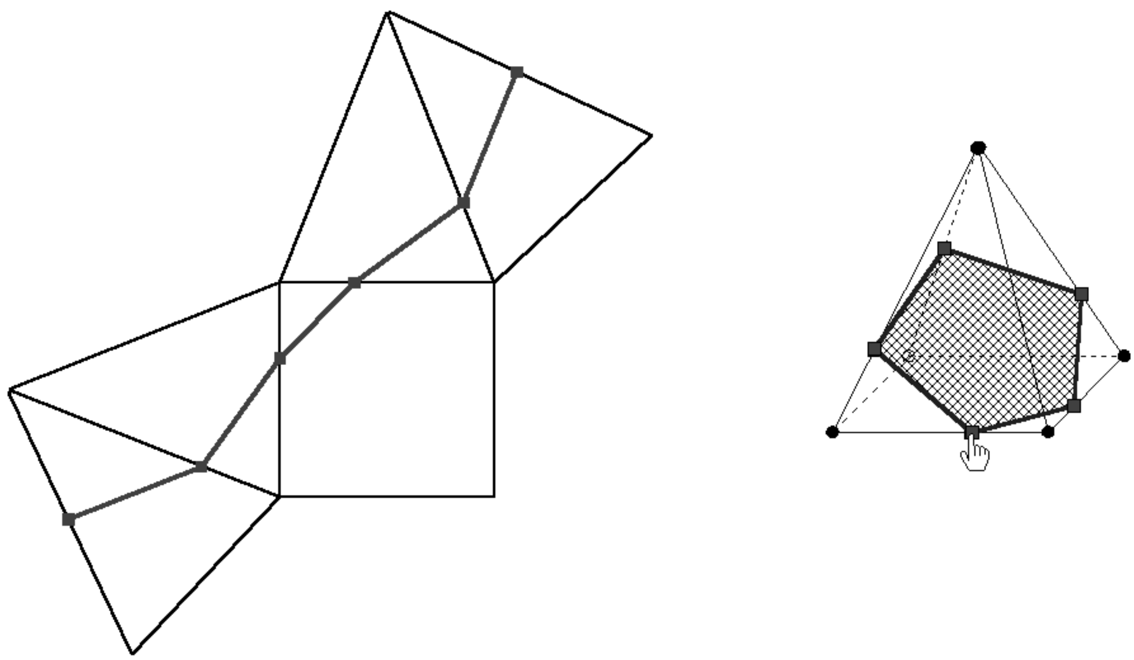


Figure 2.17.1

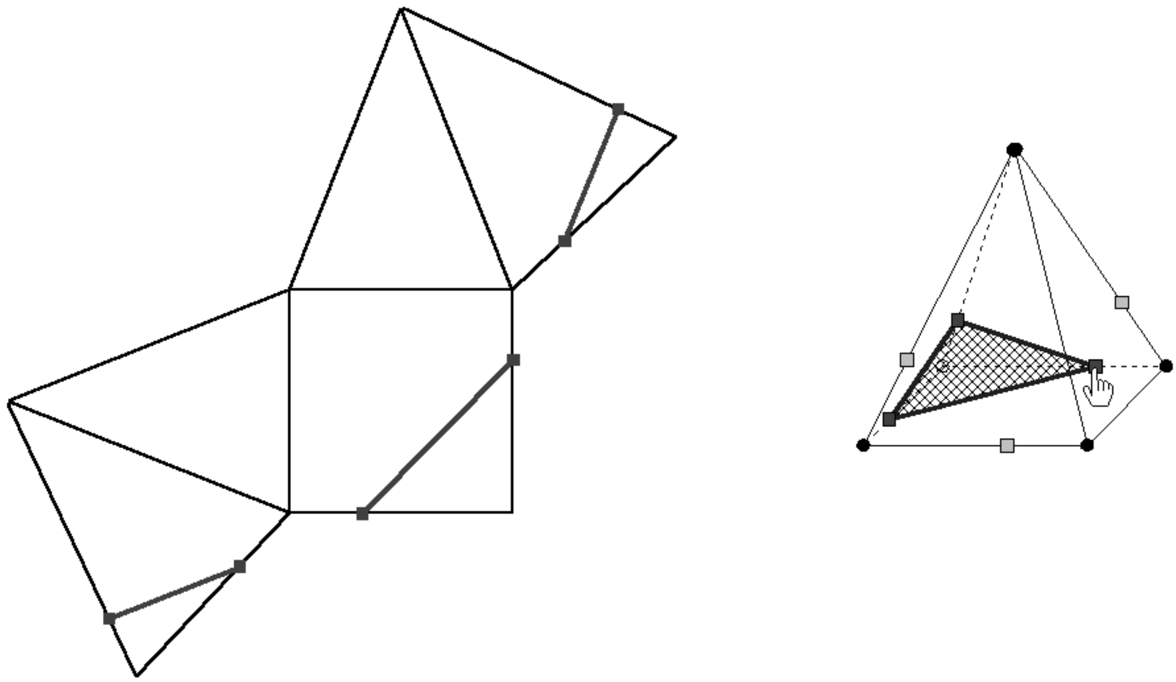


Figure 2.17.2

The variation of the polygonal forms can be visualized also by a three plane projection (Figure 2.18; side elevation, front elevation and plan).

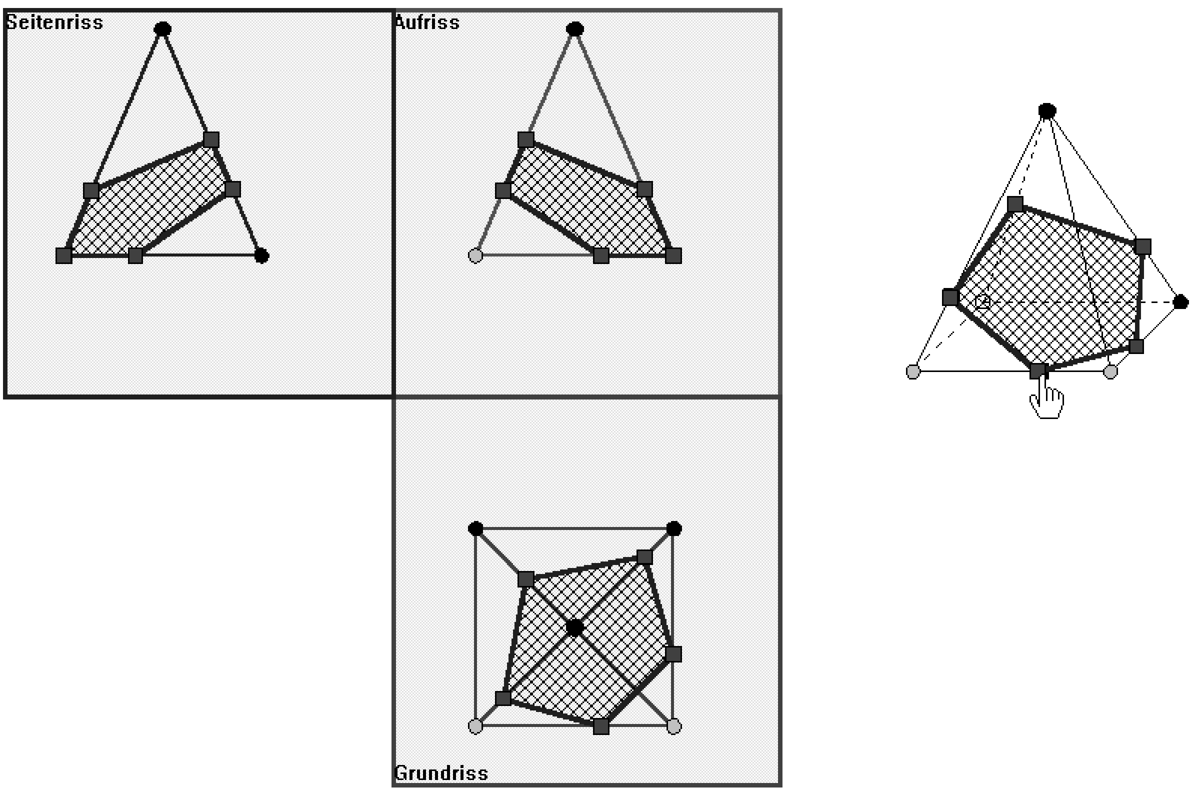


Figure 2.18

The initial pentagon can also be rotated around one of its sides to produce a quadrangle (Figure 2.19) etc.

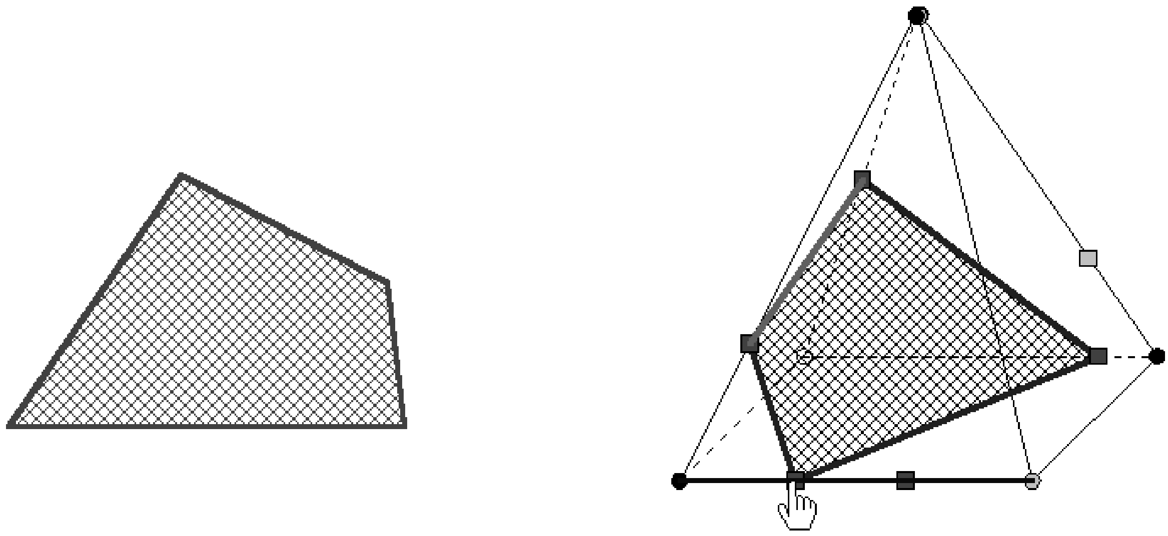


Figure 2.19

2.2.4 Particular lines in and on the pyramid

We confine ourselves to three examples of kinds of line in or on the square pyramid. We start the investigation by considering segments connecting each base vertex with the midpoint of the opposite lateral edge (Figure 2.20.1).

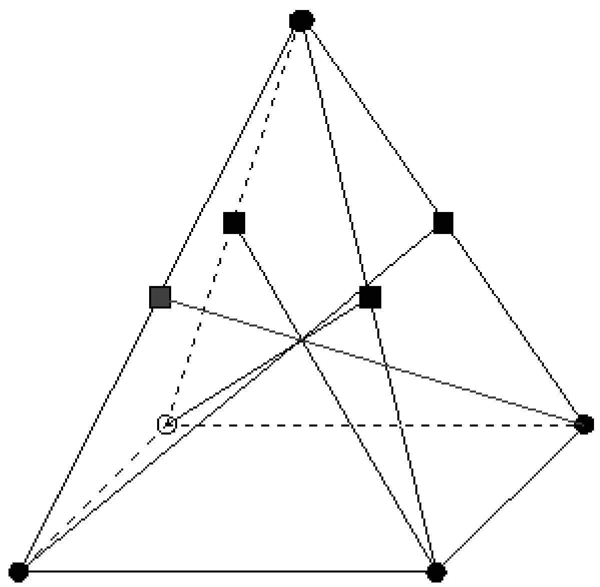


Figure 2.20.1

These lines intersect each other in one point. This property also remains unchanged if we drag the top of the pyramid (Figure 2.20.2);

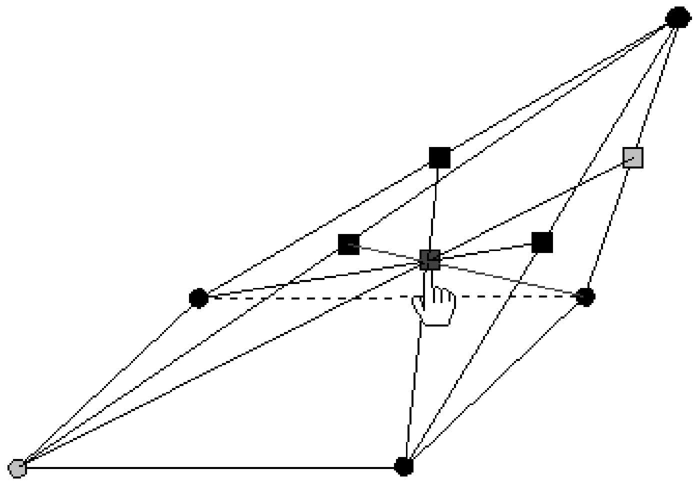


Figure 2.20.2

in addition, we can check with the option "partial point" (tool bar " Generation of solid points and line segments ") What is the ratio for the point of intersection? This ratio is invariant; its value: 2:1, measured from the base vertex. If we drop the perpendicular lines from the midpoints of the base edges onto the opposite triangular faces, then these lines intersect as long as the centre of the base of the pyramid coincides with the foot point of the altitude of the pyramide (Figure 2.21.1).

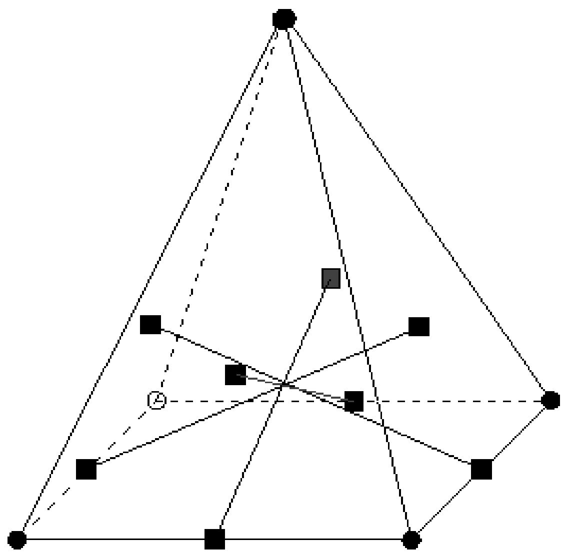


Figure 2.21.1

If this isn't the case any more, then the four lines have altogether no one common point (Figure 2.21.2).

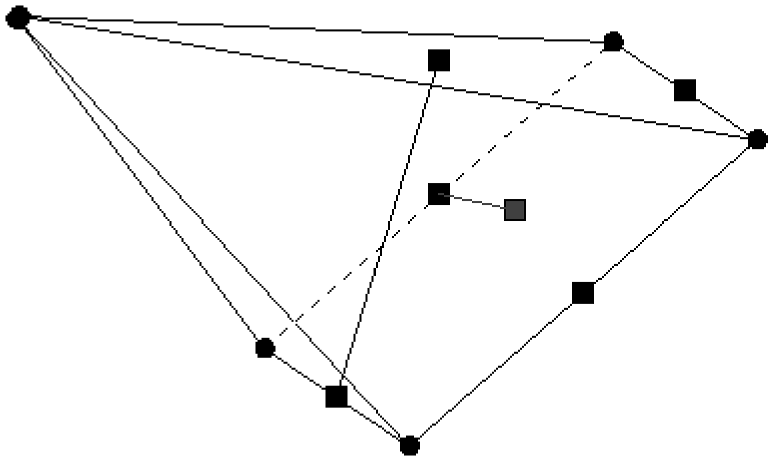


Figure 2.21.2

If we draw the centres of gravity of the triangular faces and connect them to form a quadrangle then this quadrangle has the property of being a square even if we drag the top of the pyramid (Figure 2.22.1-2.22.3)

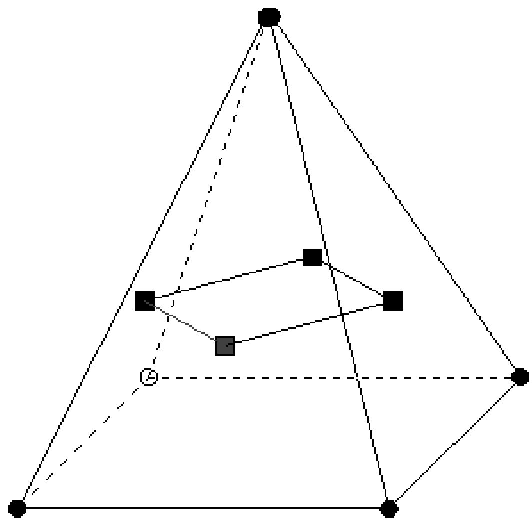


Figure 2.22.1

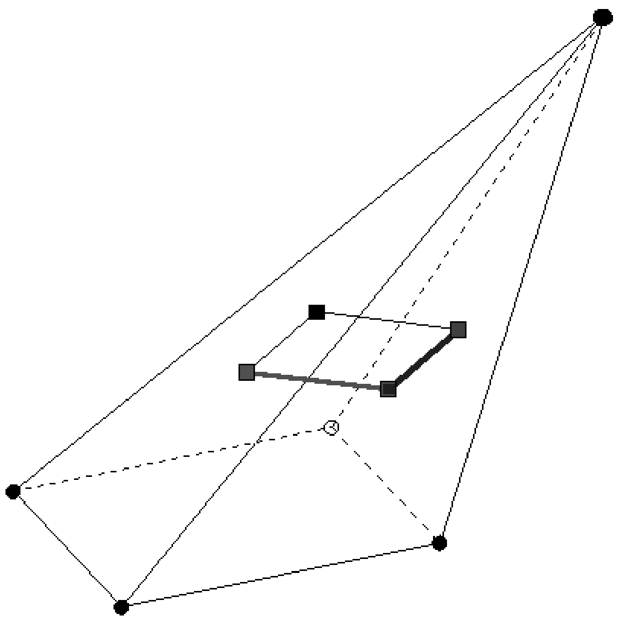


Figure 2.22.2

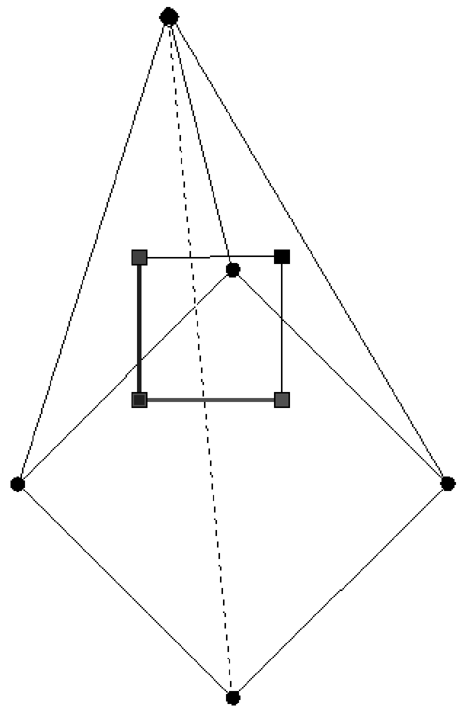


Figure 2.22.3

We can check this experimentally most simply by measuring the angle between adjacent sides and the lengths of the sides. Even if we distort the square base to any convex quadrangle the shape formed by joining the centres of gravity is invariant remaining a square ; which can be checked experimentally (Figure 2.22.4).

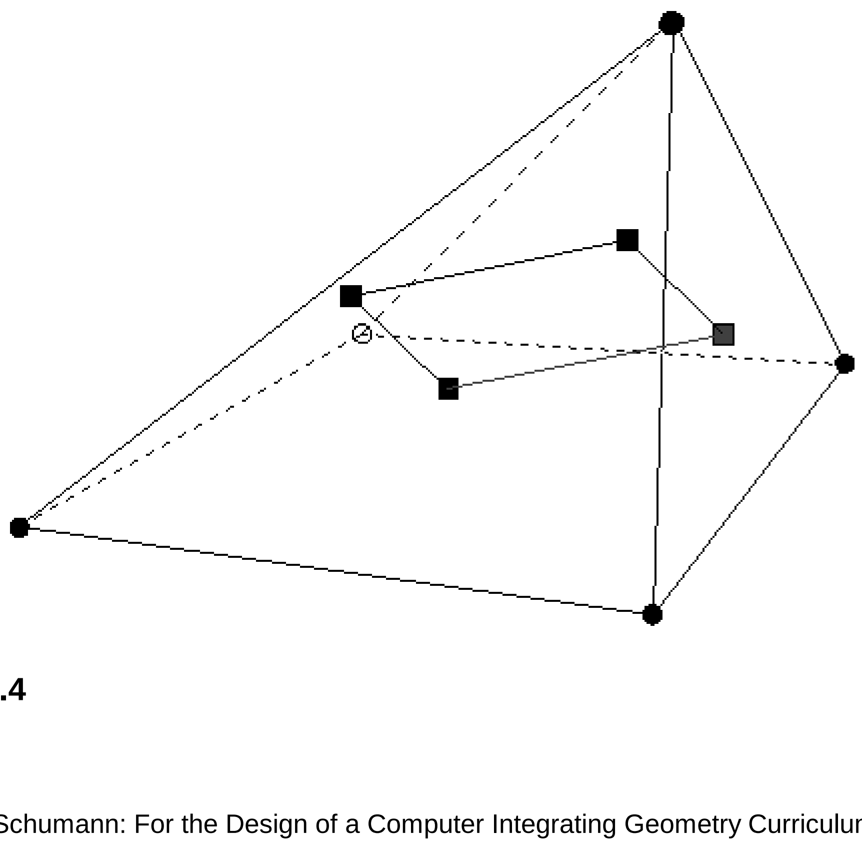


Figure 2.22.4

We close this section with an example of the shortest line (geodesic line) on the surface of a solid.

Between two points on the pyramid surface (e.g. one lies on a lateral edge, the otherone in the base, Figure 2.23.1) we can produce many connections consisting of line segments.

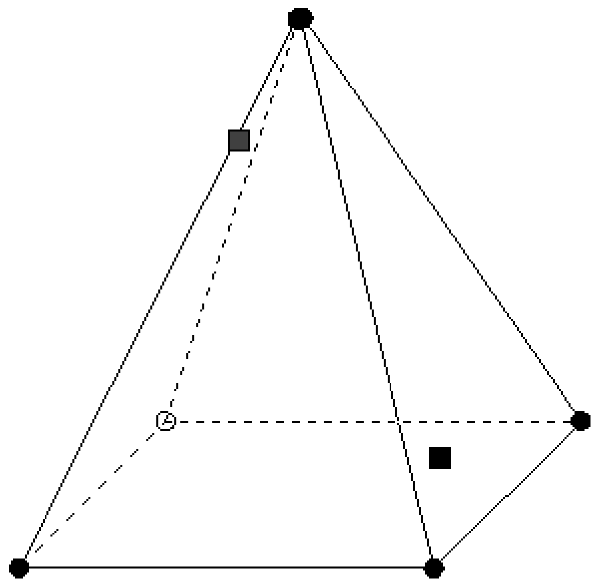


Figure 2.23.1

As an example we draw a connection consisting of three line segments (Figure 2.23.2).

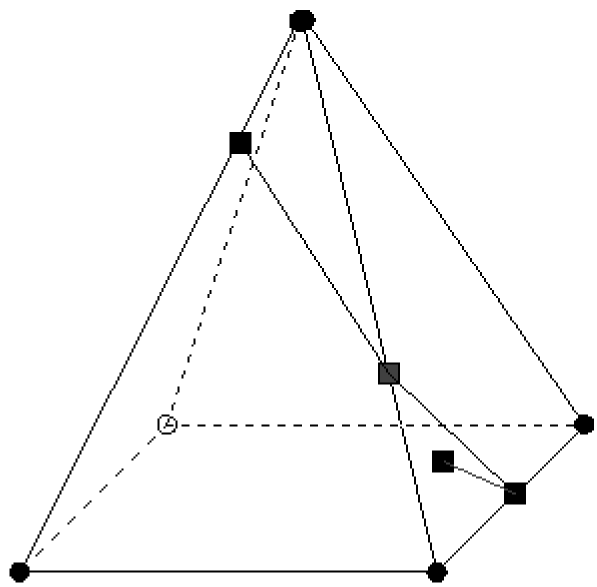


Figure 2.23.2

Which one among these unions of line segments is of minimal length?

KOERPERGEOMETRIE has a corresponding option for the construction of a path of minimal length which we invoke (Figure 2.23.3).

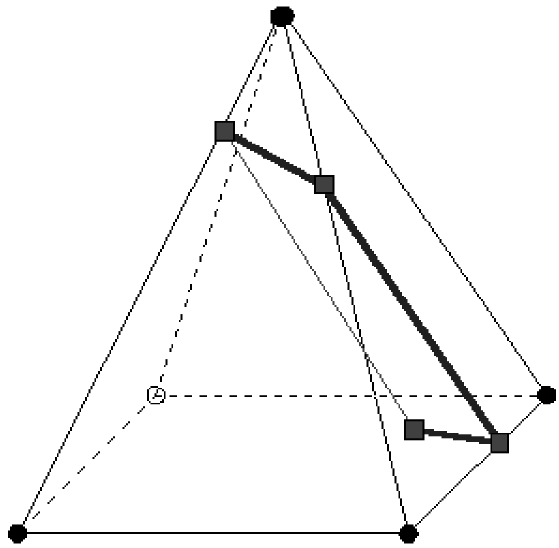


Figure 2.23.3

We check the minimal length by the fact that the union of the connecing line segments forms a straight line segment (Figure 2.23.4).

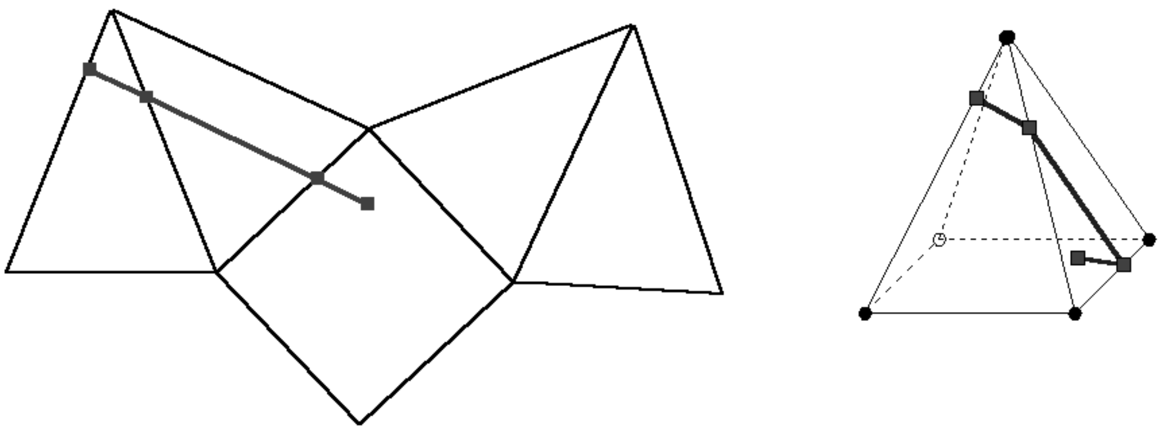


Figure 2.23.4

2.2.5 Halving pyramids

We select the section tool, which dissects a solid into two parts length ways by a plane defined by three selected points. With this option and the automatic volume measuring (surface measuring) we solve experimentally the open task: Find all typical volume bisections of the square pyramid. (Of course every volume bisection of a convex solid is a surface bisection at the same time, what as a byproduct of performing the task.)

The axis section through the midpoint of a base edge is shown (Figure 2.24.1).

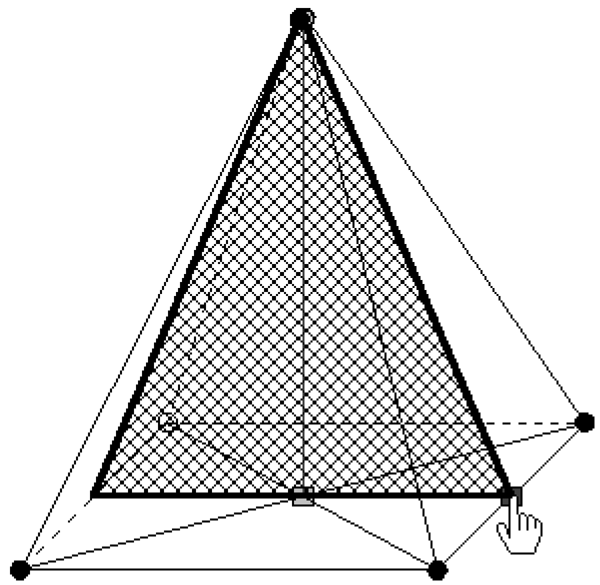


Figure 2.24.1

Result of the executed sectionis shown (Figure 2.24.2).

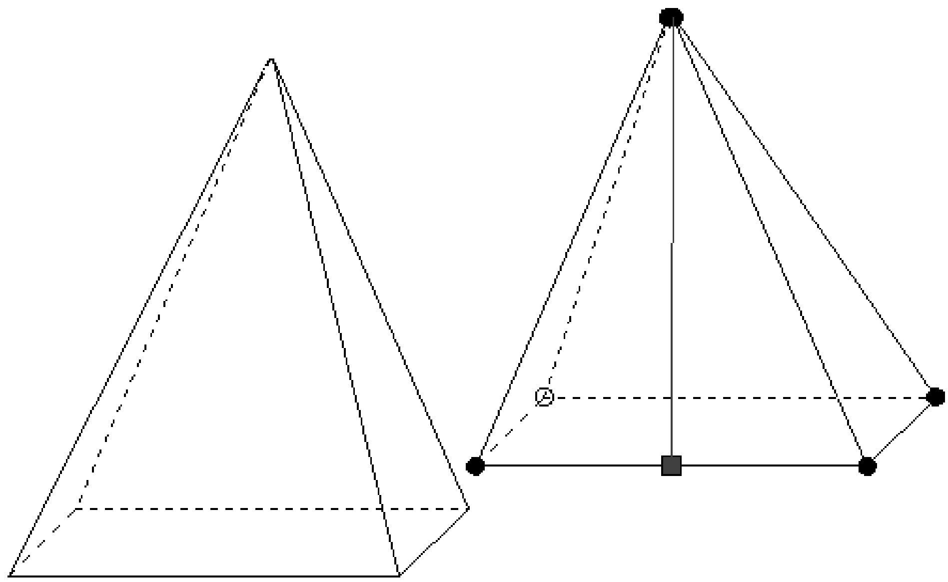


Figure 2.24.2

A further axis section is defined by a vertex of the base (Figure 2.25.1; result: Figure 2.25.2).

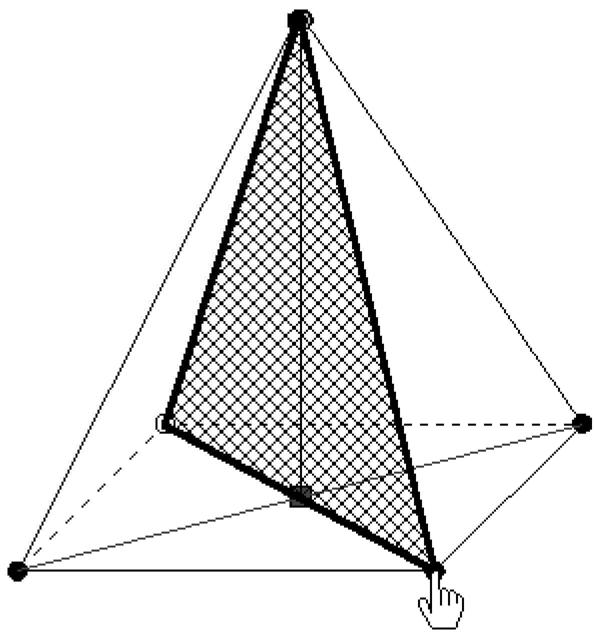


Figure 2.25.1

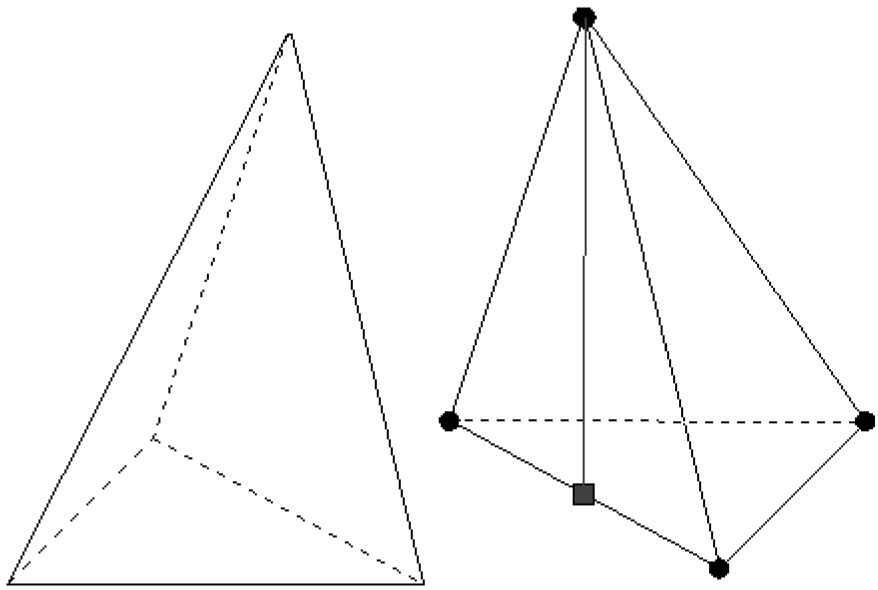


Figure 2.25.2

In the following we only give one section: Any other axis section determined by an arbitrary point on a bottom edge generates a pyramid bisection (Figure 2.26).

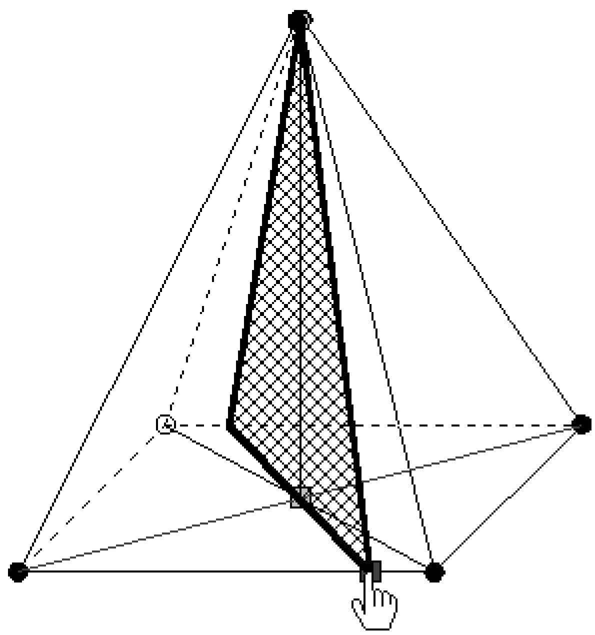


Figure 2.26

A section which contains the bottom edge and a point of an opposite lateral edge experimentally found out also halves the solid (Figure 2.27).

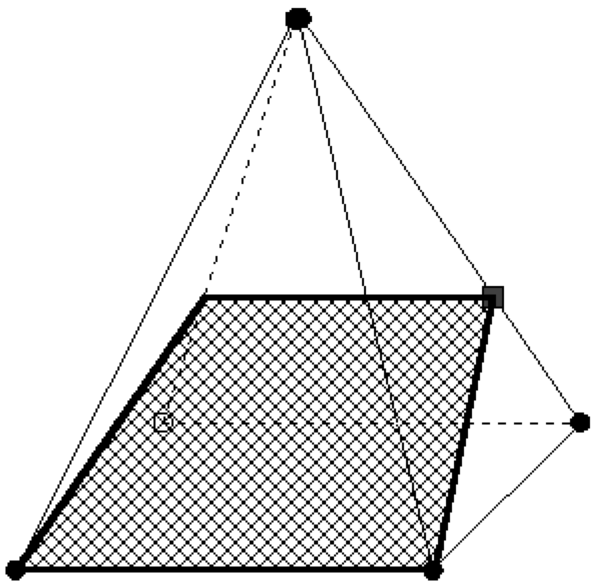


Figure 2.27

Three special points with the exception of vertices also form a halving section on different lateral edges.The following cases arise: No section line is parallel to a bottom edge (Figure 2.28);

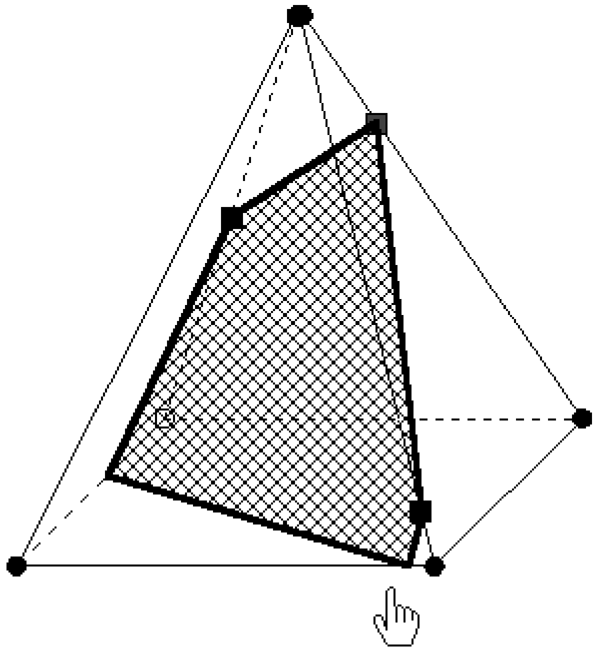


Figure 2.28

at least one section line is parallel to a bottom edge: With a section line a second is also parallel to a bottom edge (Figure 2.29),

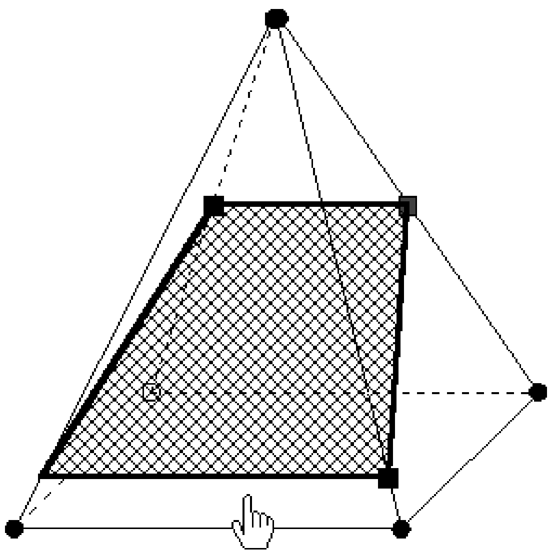


Figure 2.29

or it is parallel to the base and there is a section which bisects the square pyramid into a small square pyramid and a pyramid frustum (Figure 2.30).

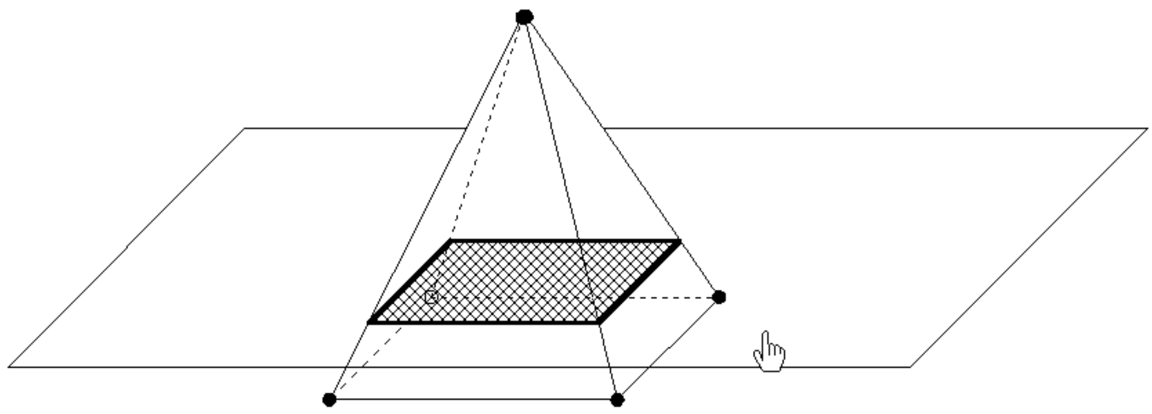


Figure 2.30

We create the frustum with the second section tool: A plane which can be dragged in parallel to the cutting plane is defined. A horizontal plane is chosen in figure 2.30. As in the preceding cases, the nets of both the solid halves can be generated (Figure 2.31.1/2).

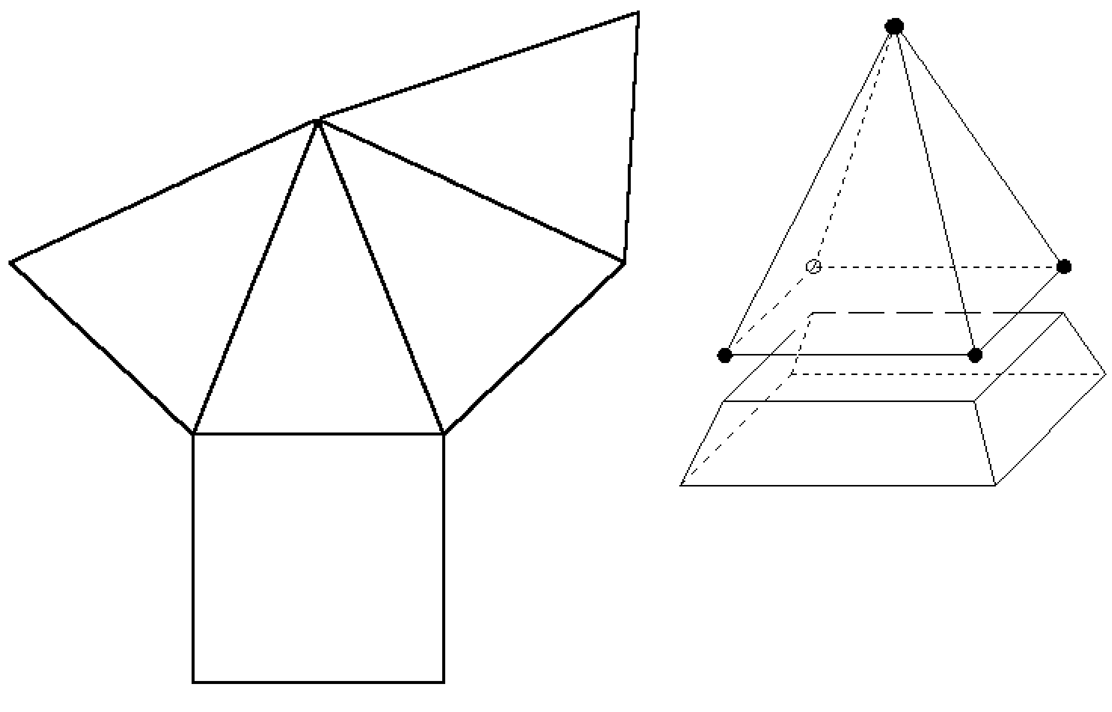
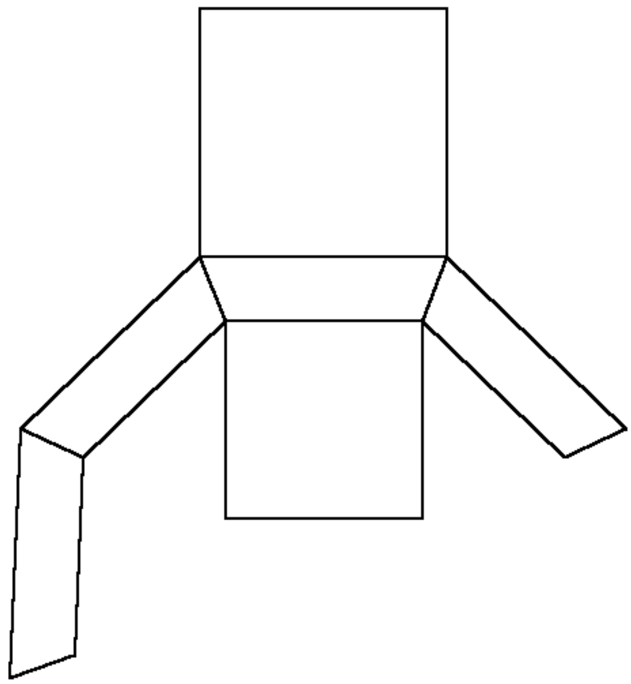


Figure 2.31.1



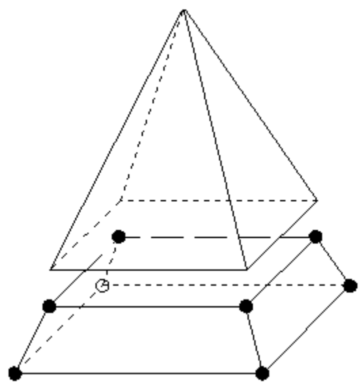


Figure 2.31.2

2.2.6 The truncated pyramid

The Babylonian derivation of the volume formula for the pyramid frustum (Figure 2.32.1) created as in figure 2.30 presupposes its reduction in parts for which the volume formulae are known already. We use the second section tool to execute suitable sections with movable vertical sections (Figure 2.32.2).

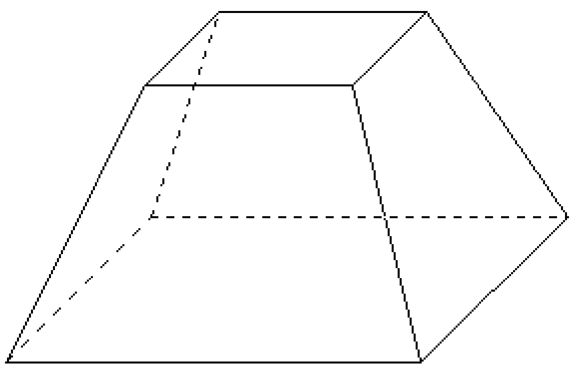


Figure 2.32.1

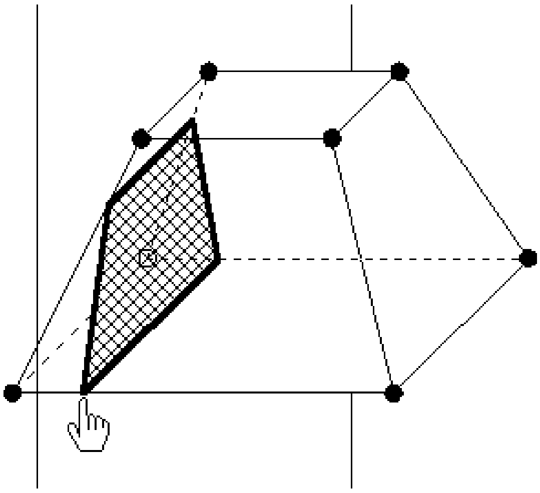


Figure 2.32.2

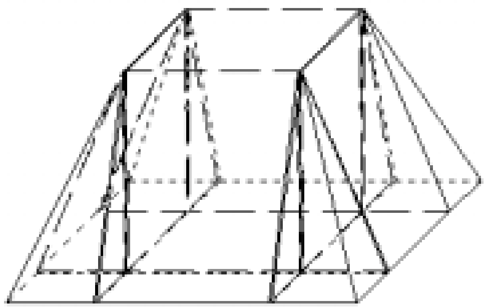


Figure 2.32.3

The figure 2.32.3 shows the pyramid frustum with the corresponding section lines, on which we look at different positions by solid rotations (Figure 2.32.4/5).

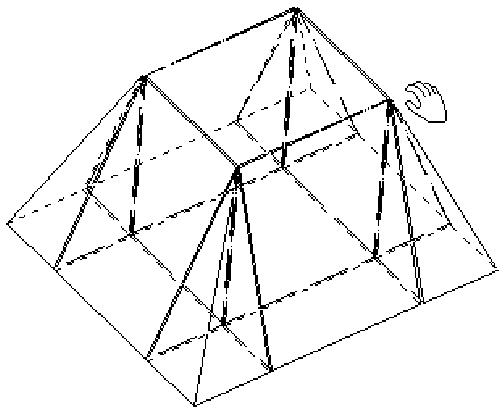


Figure 2.32.4

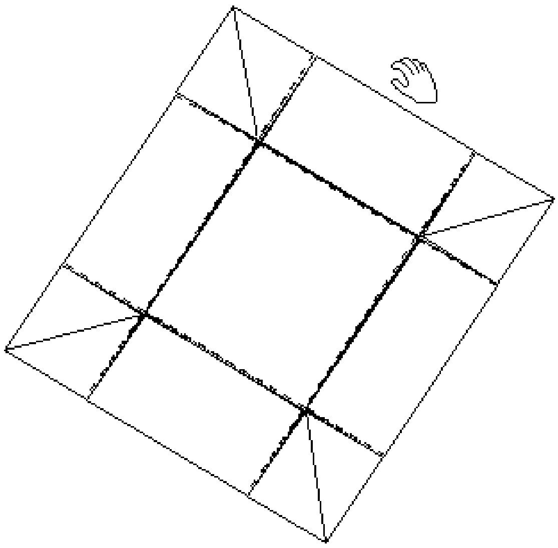


Figure 2.32.5

We drag the parts away from each other (Figure 2.32.6),

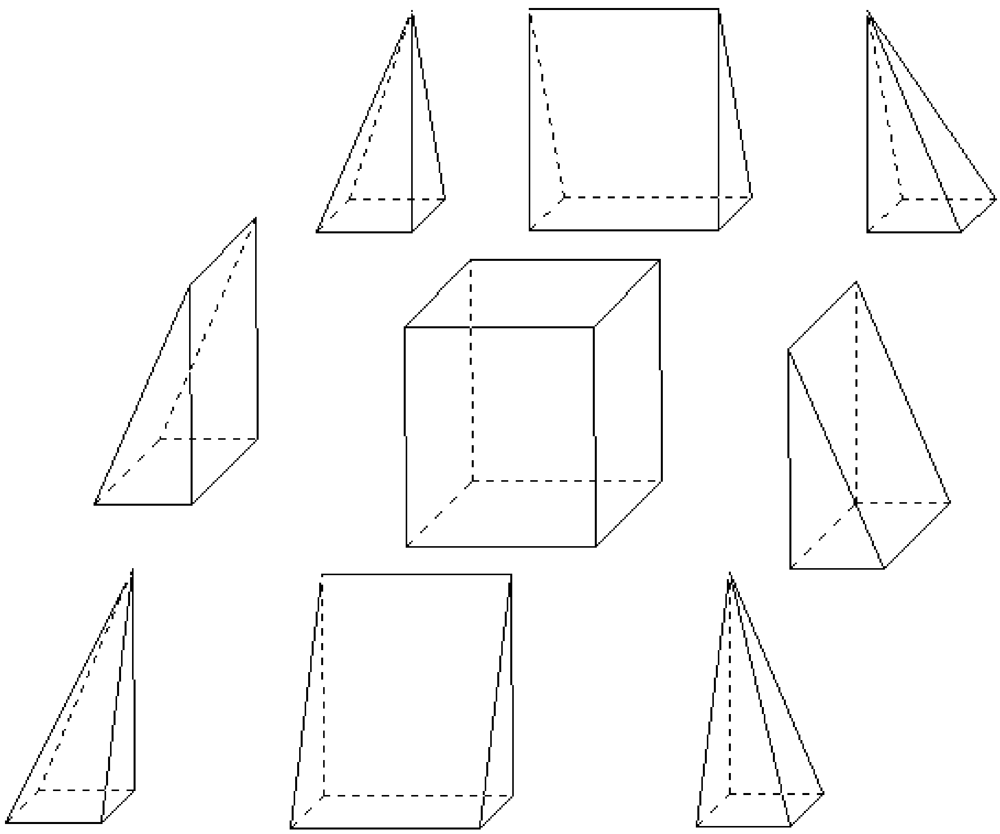


Figure 2.32.6

and join them together to form a square pyramid and two trihedral prisms; a square cuboid is left (Figure 2.32.7).

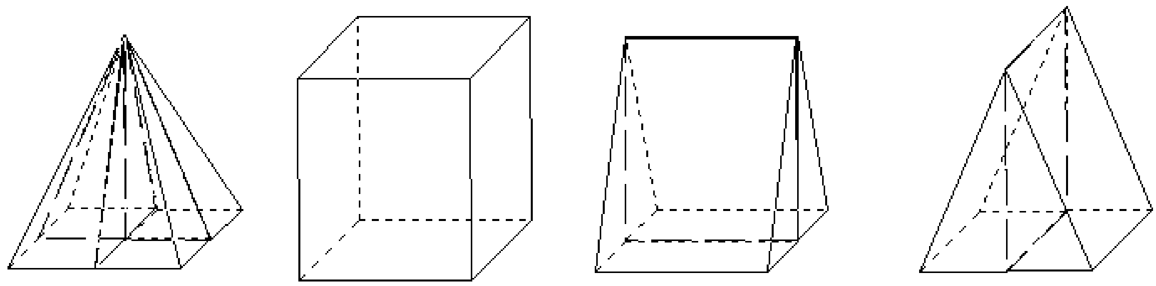


Figure 2.32.7

By chosing suitable rather fiddly rotations around axes we get a cuboid from the two prisms (Figure 2.32.8).

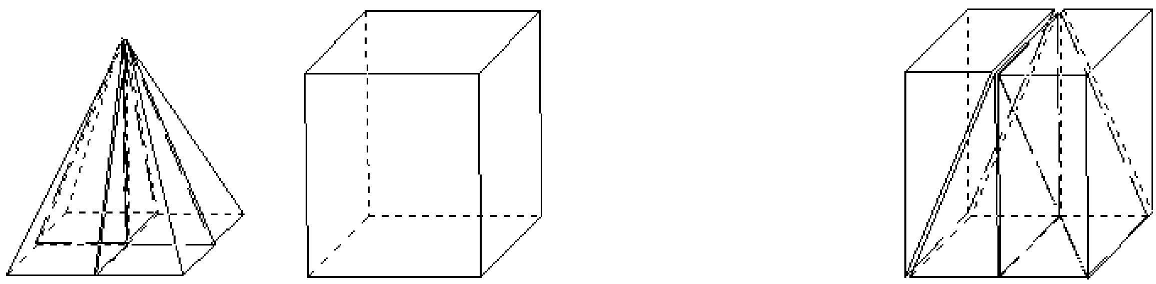


Figure 2.32.8

Now we can use the known volume formulae to derive the formula for the frustum.

2.2.4 Cutting off pyramids

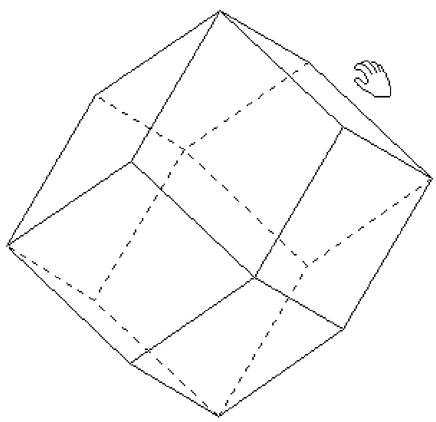


Figure 2.33.1

The figure 2.33.1 shows a dodecahedron consisting of mutually congruent rhombs. Therefore one also calls this solid a rhomb dodecahedron. Its vertices are formed from three or four rhombs; therefore it isn't a regular solid.

It can be measured with the measuring tools of KOERPERGEOMETRIE. A cube can be inscribed into this solid (Figure 2.33.2).

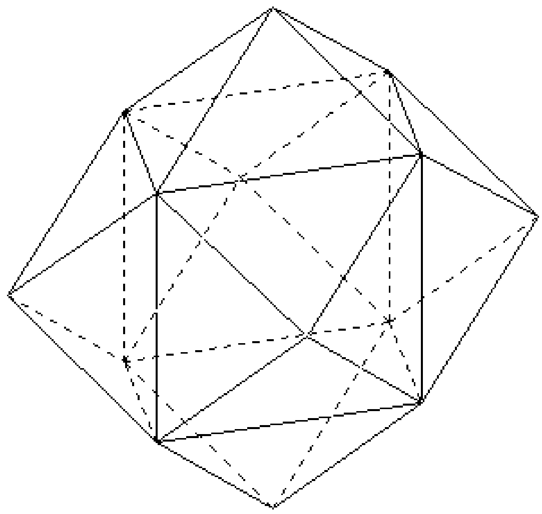


Figure 2.33.2

The rhomb dodecahedron can be combined from four square pyramids and a cube. We cut off one of these pyramids (Figure 2.33.3) and define its base as a plane for reflection (Figure 2.33.4) and reflect the pyramid.

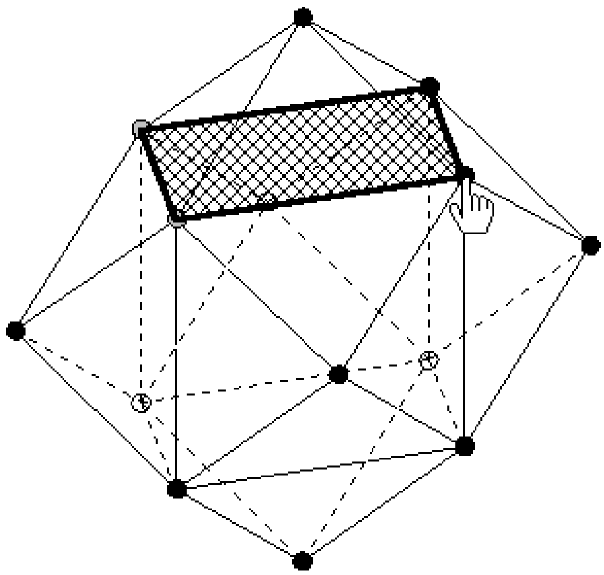


Figure 2.33.3

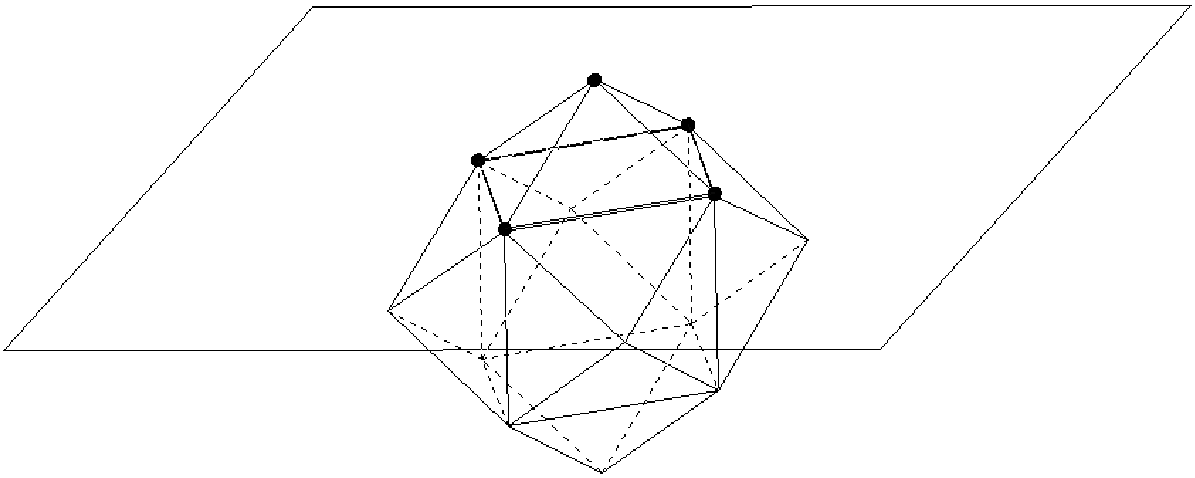


Figure 2.33.4

The top of the mirrored pyramid coincides with the centre of the cube (Figure 2.33.5);

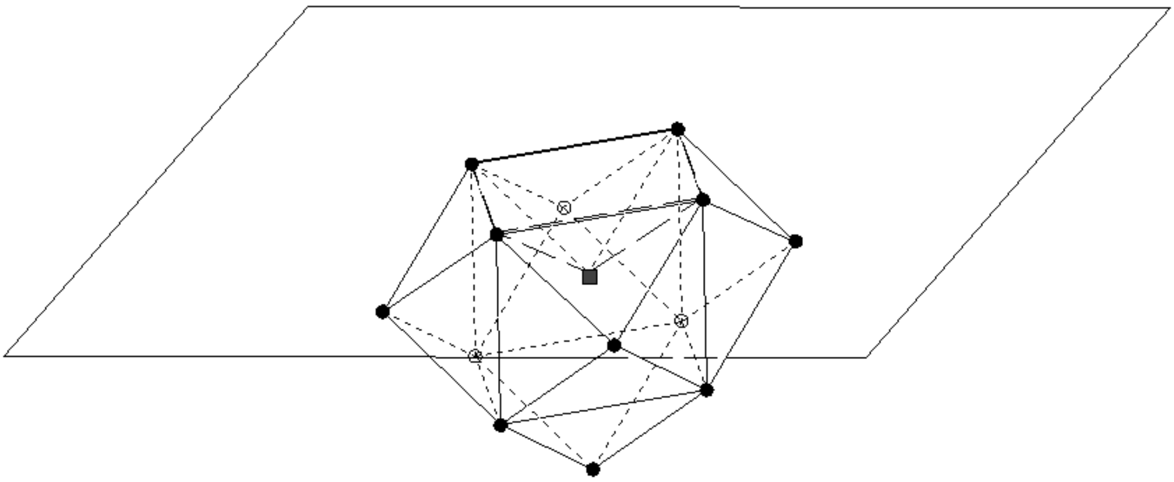


Figure 2.33.5

the pyramid height is half as long as the cube edge.

About the implementation of solids by coordinates: The input of the rhomb dodecahedron starts from the vertices of a unit cube, upon which we put twelve suitable rhombs. One can create the following programs for the two solids with any text editor.

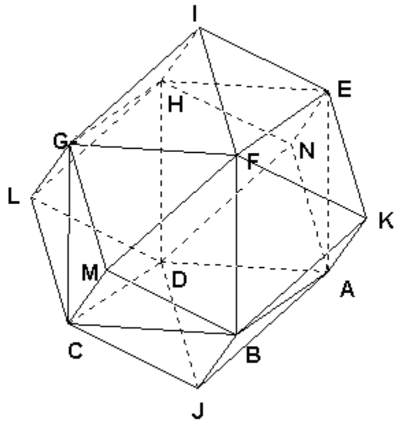
Program for the rhomb dodecahedron with inscribed cube:

1,E, 1.0,-1.0, 0.0,A
1,E, 1.0, 0.0, 0.0,B
1,E, 0.0, 0.0, 0.0,C
1,E, 0.0,-1.0, 0.0,D
1,E, 1.0,-1.0, 1.0,E
1,E, 1.0, 0.0, 1.0,F
1,E, 0.0, 0.0, 1.0,G
1,E, 0.0,-1.0, 1.0,H
1,E, 0.5,-0.5, 1.5,I
1,E, 0.5,-0.5,-0.5,J
1,E, 1.5,-0.5, 0.5,K
1,E,-0.5,-0.5, 0.5,L
1,E, 0.5, 0.5, 0.5,M
1,E, 0.5,-1.5, 0.5,N
1,F,EFI
1,F,FGI
1,F,GHI
1,F,HEI
1,F,BAJ
1,F,CBJ
1,F,DCJ
1,F,ADJ
1,F,ABK
1,F,BFK
1,F,FEK
1,F,EAK
1,F,CDL
1,F,DHL
1,F,HGL
1,F,GCL
1,F,BCM
1,F,CGM
1,F,GFM
1,F,FBM
1,F,DAN
1,F,AEN
1,F,EHN
1,F,HDN

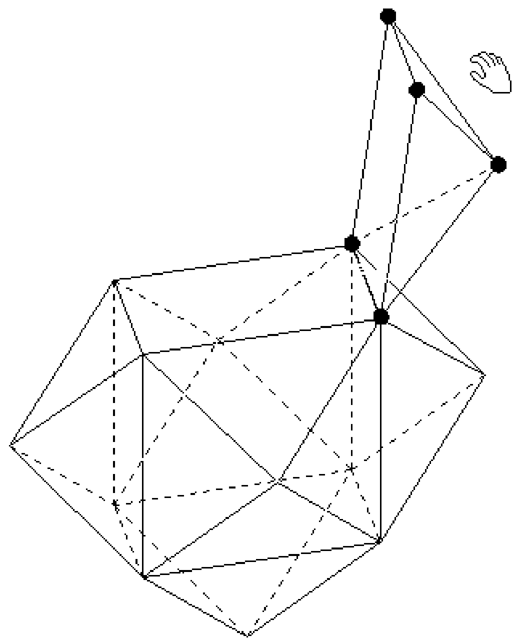
Remarks: 1stands for the solid to be defined (here only for a subsolid), E (F) for the vertices to be defined, (faces); A, B, C, D, F, G, H are the denotions for the vertices of the cubeI, I,J,K,L,M,N for the tops of the pyramids.
C forms the coordinate origin; the inscribed cube is assumed to be a unit cube.

Program for the rhomb dodecahedron:

1,E, 1.0,-1.0, 0.0,A
1,E, 1.0, 0.0, 0.0,B
1,E, 0.0, 0.0, 0.0,C
1,E, 0.0,-1.0, 0.0,D
1,E, 1.0,-1.0, 1.0,E
1,E, 1.0, 0.0, 1.0,F
1,E, 0.0, 0.0, 1.0,G
1,E, 0.0,-1.0, 1.0,H
1,E, 0.5,-0.5, 1.5,I
1,E, 0.5,-0.5,-0.5,J
1,E, 1.5,-0.5, 0.5,K
1,E,-0.5,-0.5, 0.5,L
1,E, 0.5, 0.5, 0.5,M
1,E, 0.5,-1.5, 0.5,N
1,F,KFIE
1,F,MGIF
1,F,LHIG
1,F,NEIH
1,F,KBMF
1,F,MCLG
1,F,LDNH
1,F,NAKE
1,F,KAJB
1,F,MBJC
1,F,LCJD
1,F,NDJA



Therefore the volume of the rhomb dodecahedron is twice that of the cube. We also can see this by the fact that we form a cube which is congruent to the inscribed out of the six pyramids put on.To this we cut off one by one the four pyramids and turn these suitably around one of the bottom edges (Figure 2.34.1/2) and move the fifth into the gap left (Figure 2.34.3).



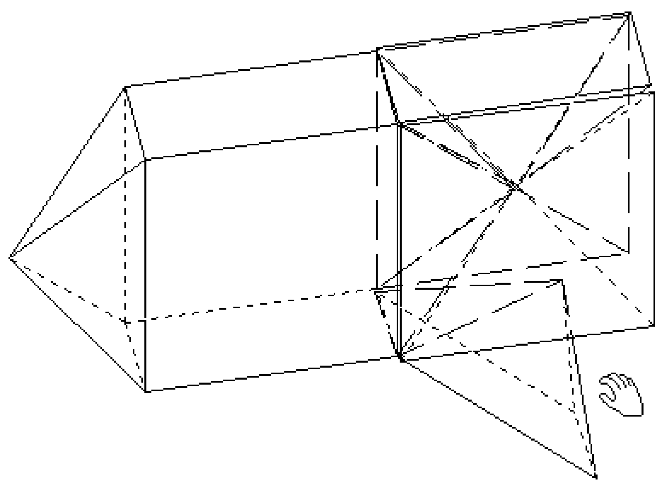


Figure 2.34.1/2

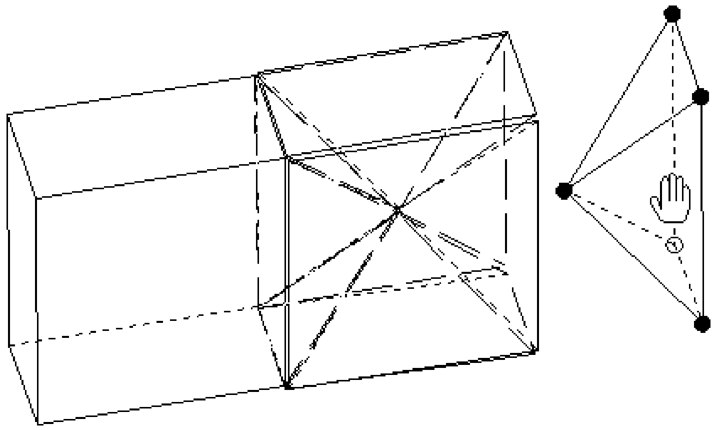


Figure 2.34.3

A further example of cutting off pyramids:
We define a four-sided double pyramid with equal edges (regular octahedron), cut off its six corners up to the midpoints of the edges and get a solid with six squares and eight equilateral triangles as faces (cubo octahedron).

2.2.8 Cutting out pyramids

Square pyramids can be cut out of solids. A square cuboid is among others suitable for this: First we draw the pyramid inside the square cuboid (Figure 2.35.1), we then make cuts as in figure 2.35.2 and 2.35.3 using the first cutting tool.

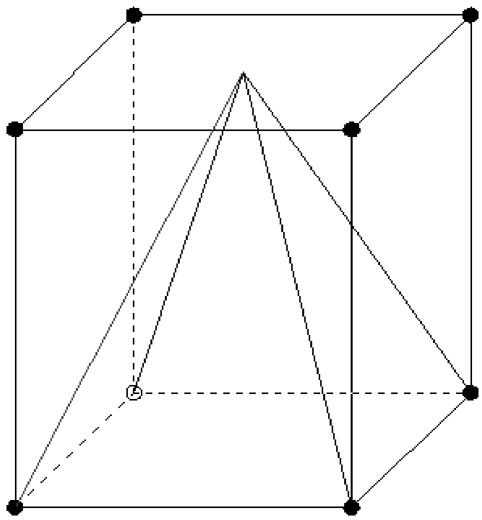


Figure 2.35.1

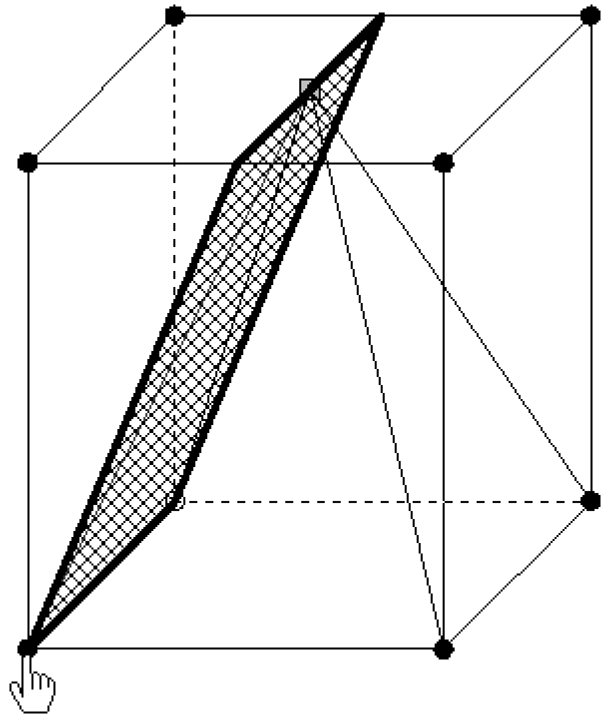


Figure 2.35.1

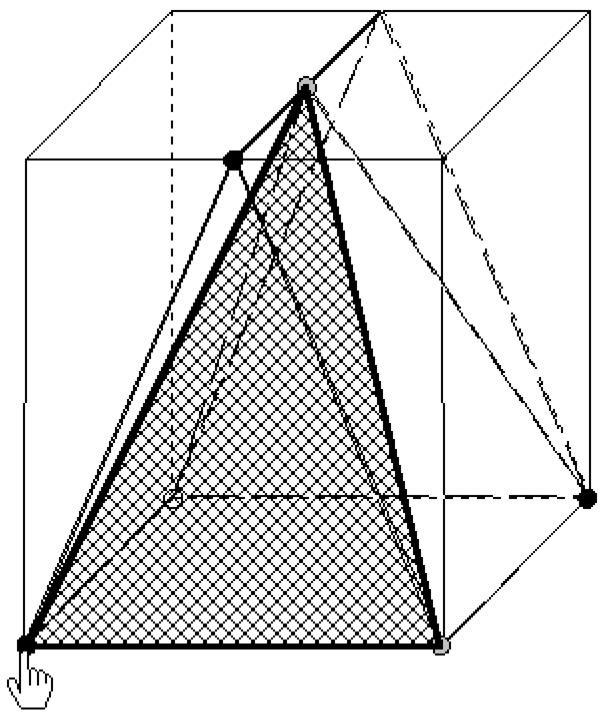


Figure 2.35.3

We get four "scrap solids" (Figure 2.35.4).

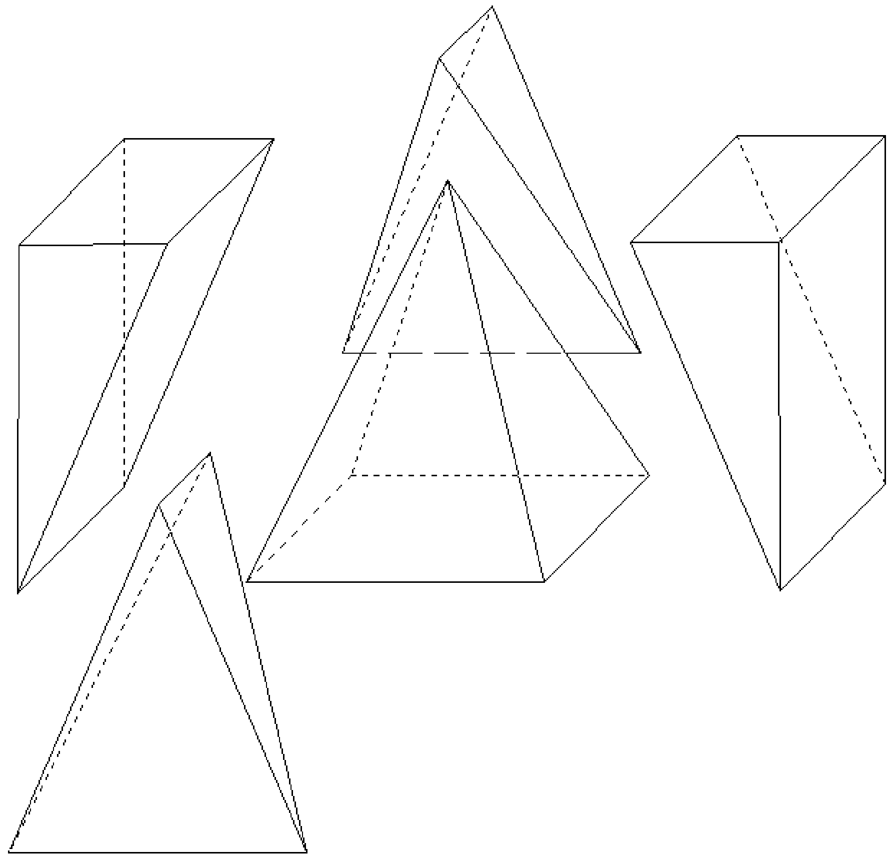


Figure 2.35.4

If we use the second section tool, then we get a different solution (Figure 2.36.1/2);

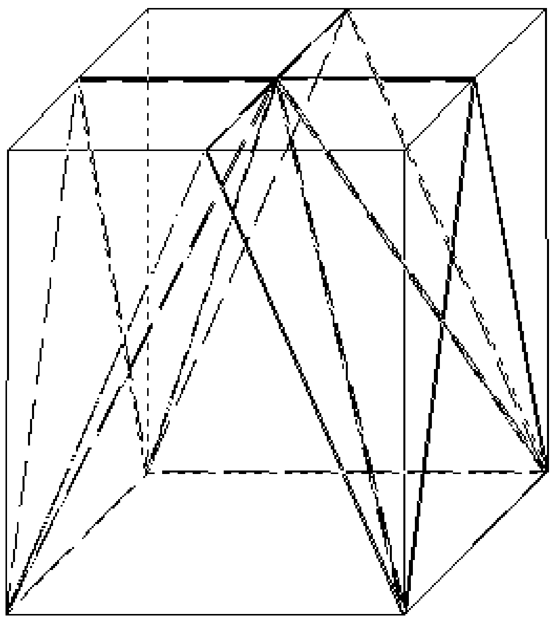


Figure 2.36.1

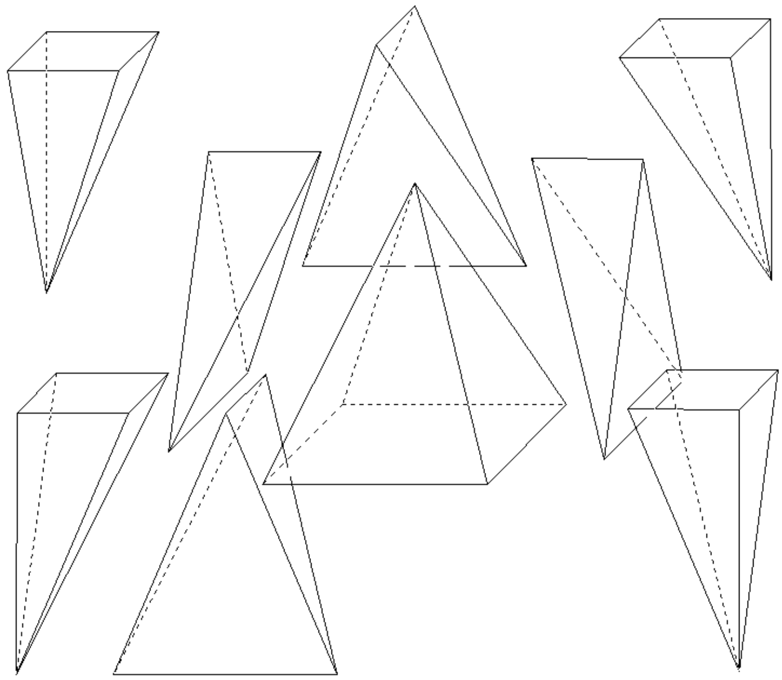


Figure 2.36.2

If we partially drag off the generated subsolids then we get an intersection of two roofs (Figure 2.36.3).

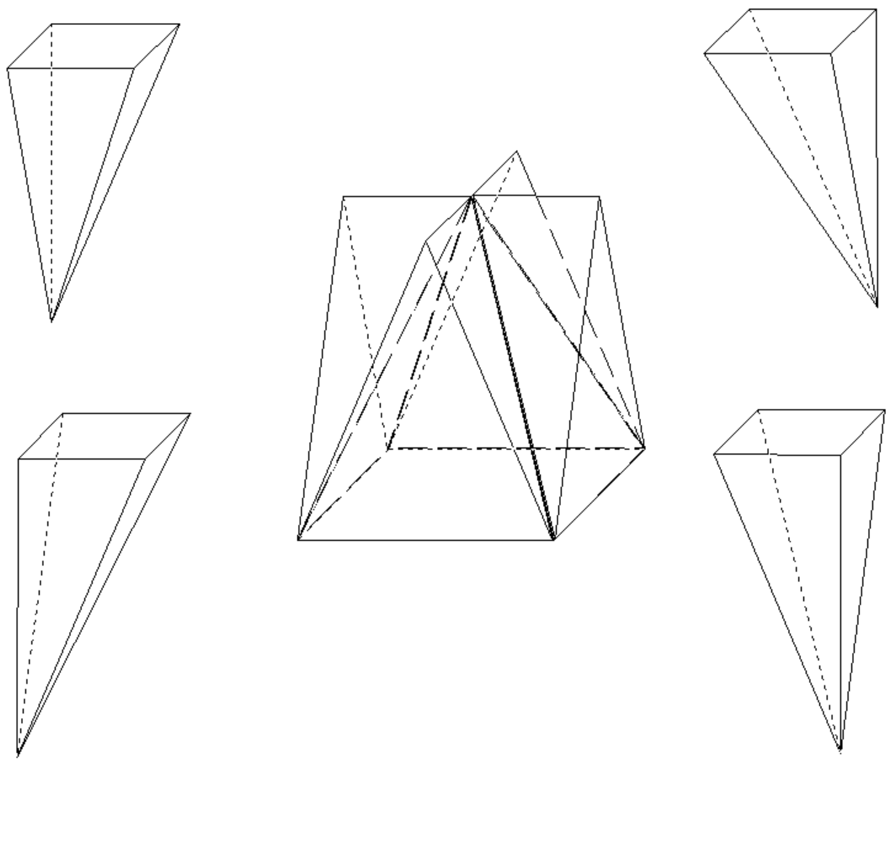


Figure 2.36.3

We can dissect also the cube into pyramids. We demonstrate only one of the possibilities here. As result we get three mutually congruent pyramids with square bases; unfortunately one of the pyramids is always dissected into two trihedral pyramids (Figure 2.37.1/2).

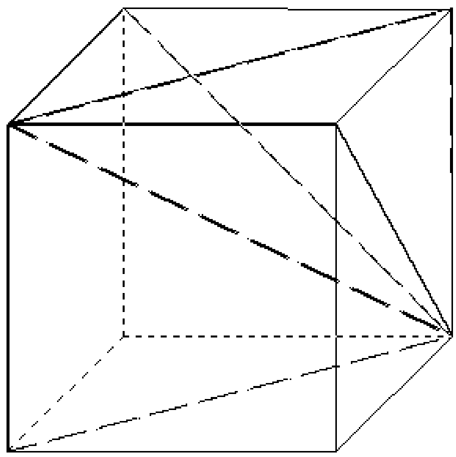


Figure 2.37.1

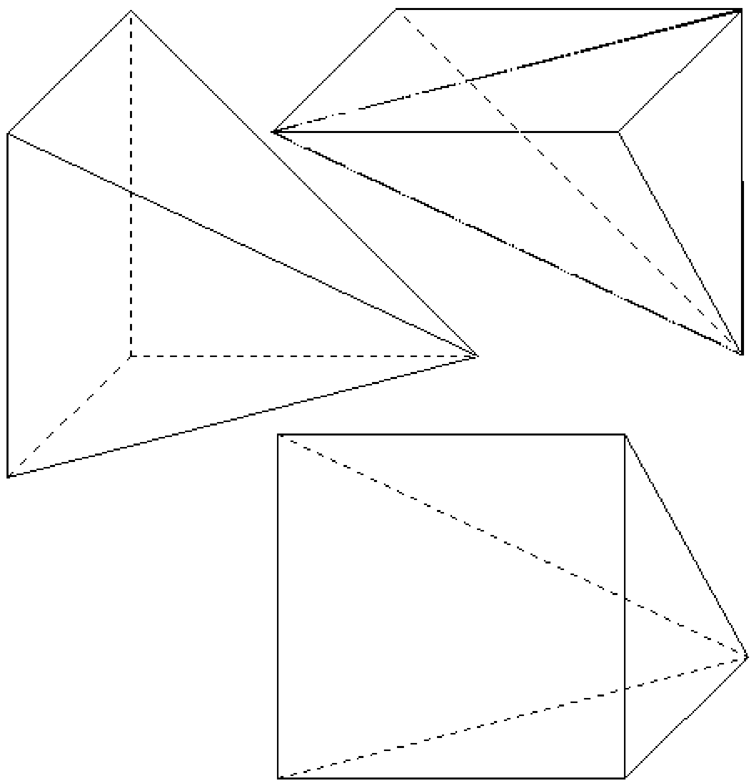


Figure 2.37.2

The net of such a pyramid shows the true form of the faces to be described (Figure 2.37.3).

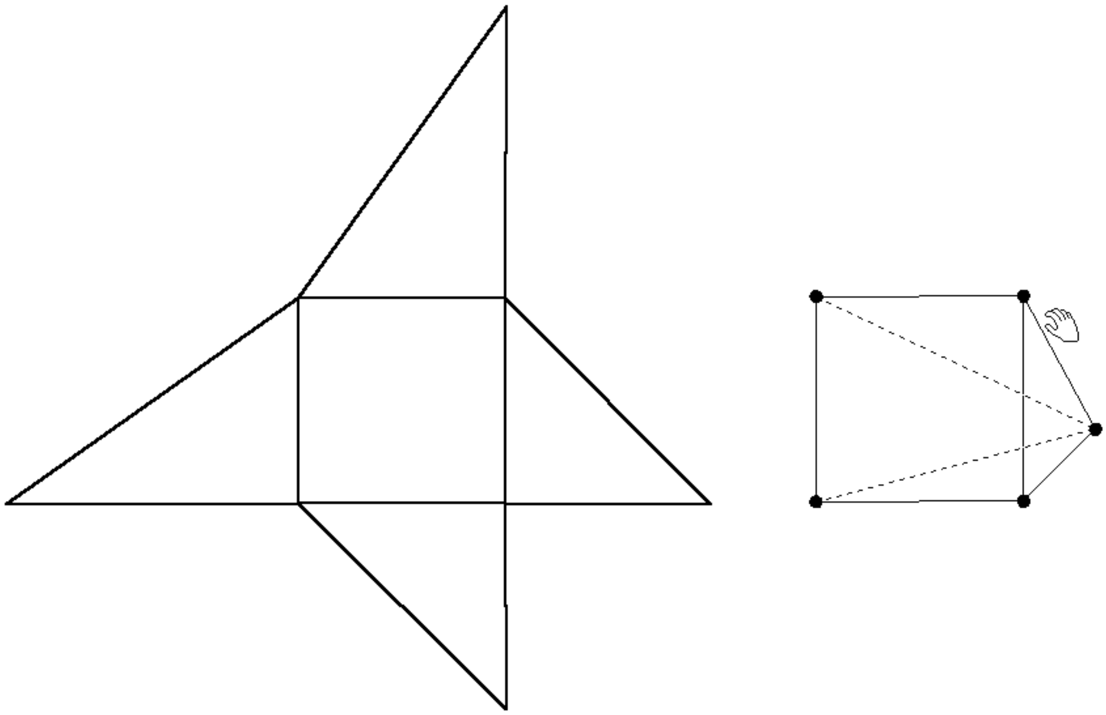


Figure 2.37.3

2.2.9 Pyramid dissections into various solids

A double pyramid is to be cut out of a square pyramid. Figure 2.38.1 shows the project for this.

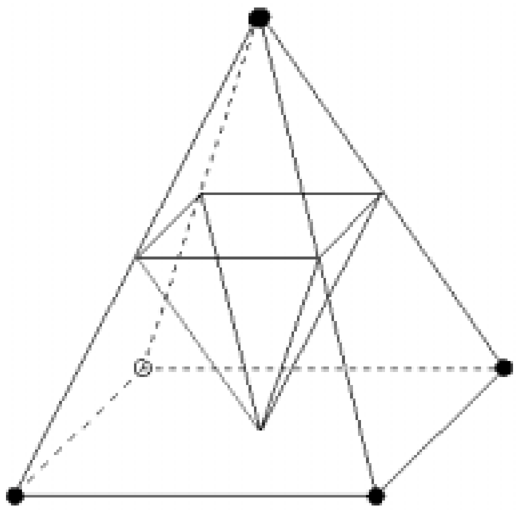


Figure 2.38.1

With the first section tool we get a dissection as in figure 2.38.2.

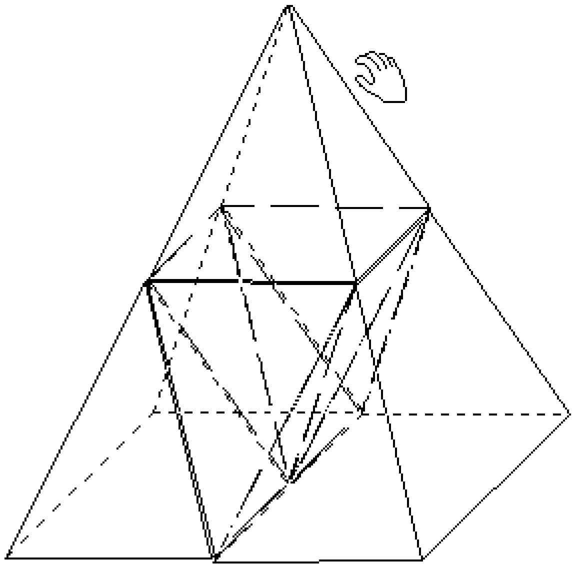


Figure 2.38.2

We have a look at this dissection by rotating the solid (Figure 2.38.3).

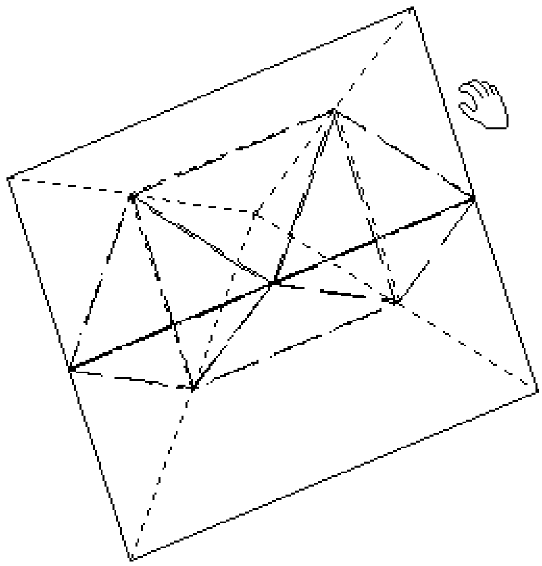


Figure 2.38.3

The result of the dragging apart shows the "exploded view" in figure 2.38.4.

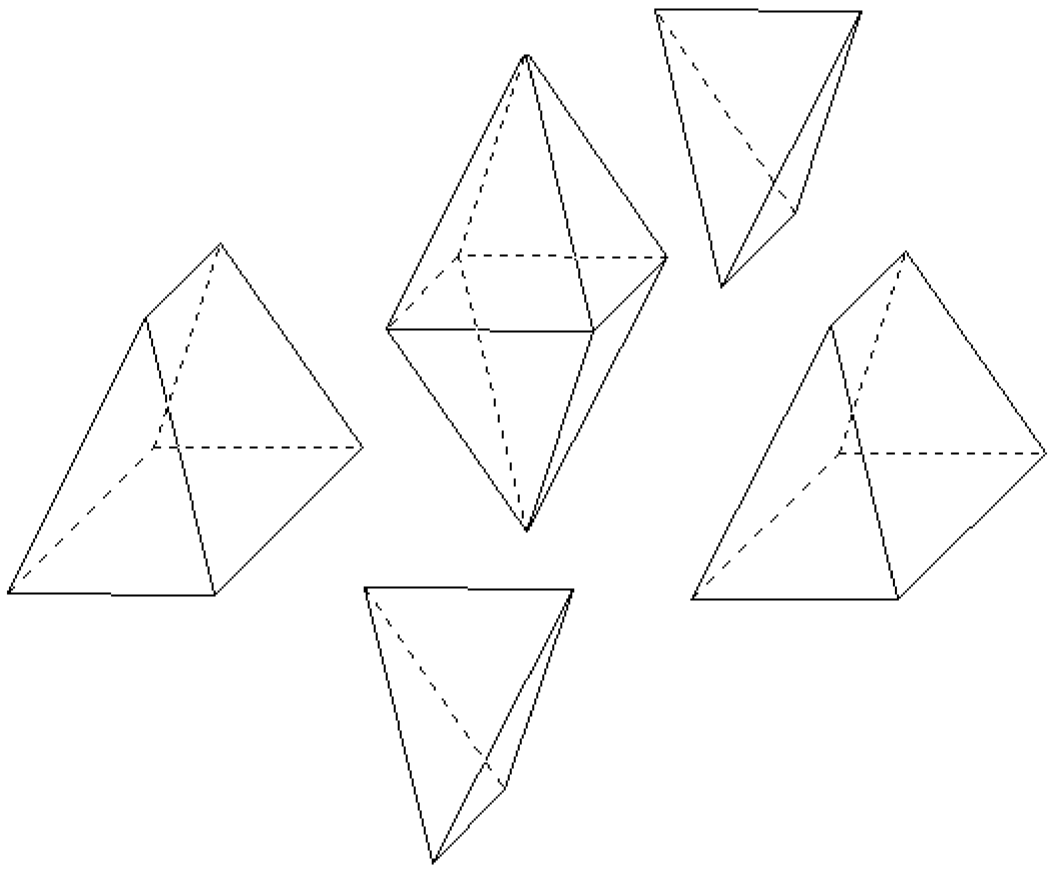


Figure 2.38.4

The whole section scene can be rotated around. The parts form a "pyramid jigsaw puzzle". The net of the octahedron can be seen in figure 38.5; the jigsaw puzzle parts are also made available by printout for tactile perception.

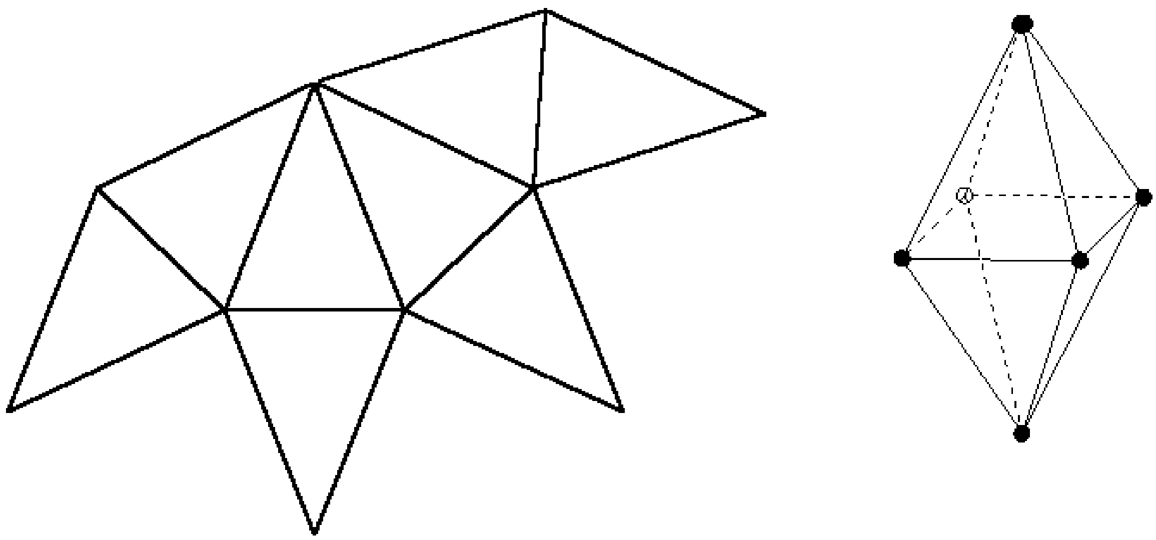


Figure 2.38.5

Production of an "irregular" jigsaw puzzle: For example we put two horizontal and two vertical sections in a square pyramid (Figure 2.39.1).

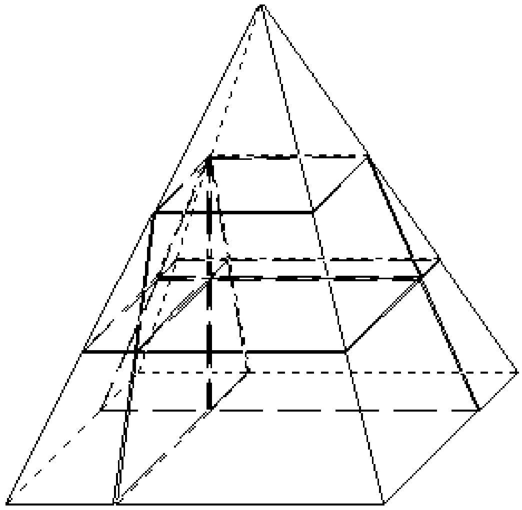


Figure 2.39.1

We seperate the parts from each other and mix them randomly (Figure 2.39.2).

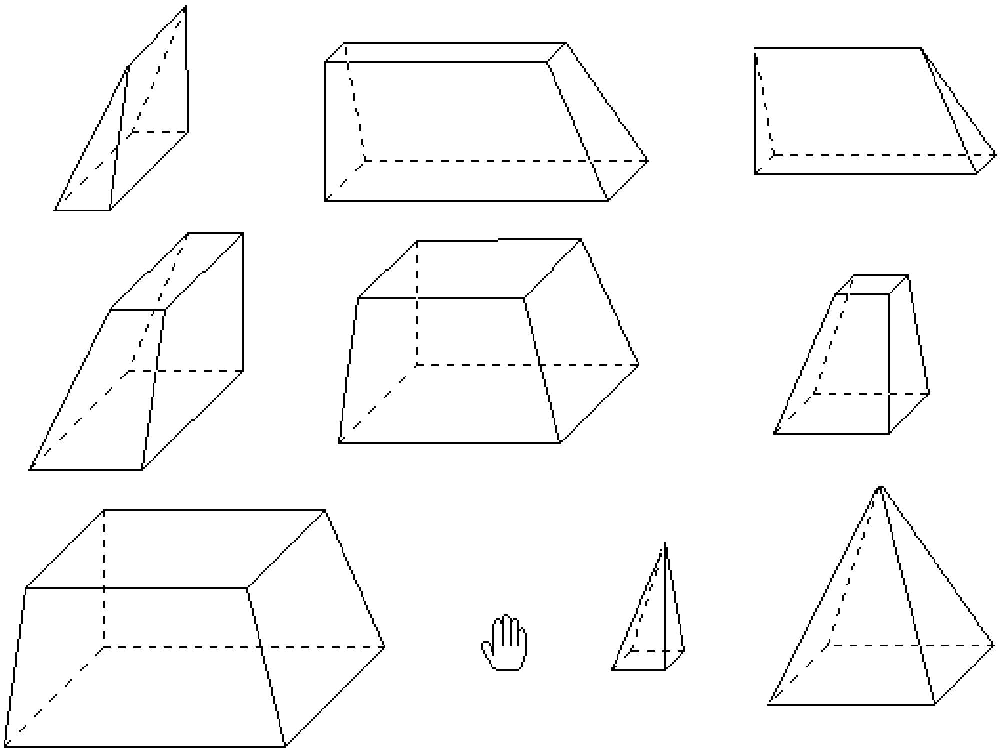


Figure 2.39.2

It is a challenging task to fit together these seperate parts to form a square pyramid.

2.2.10 Fitting together pyramids with various solids

We can combine a pyramid with itself and with different basic solids; three such results of combination show figures 2.40.1 – 2.40.3.

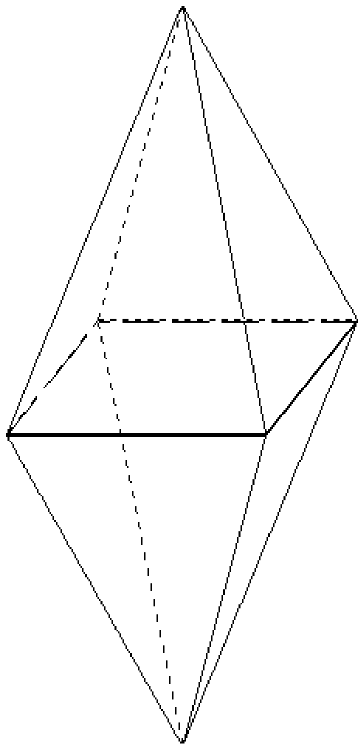


Figure 2.40.1 (Pyramid with pyramid)

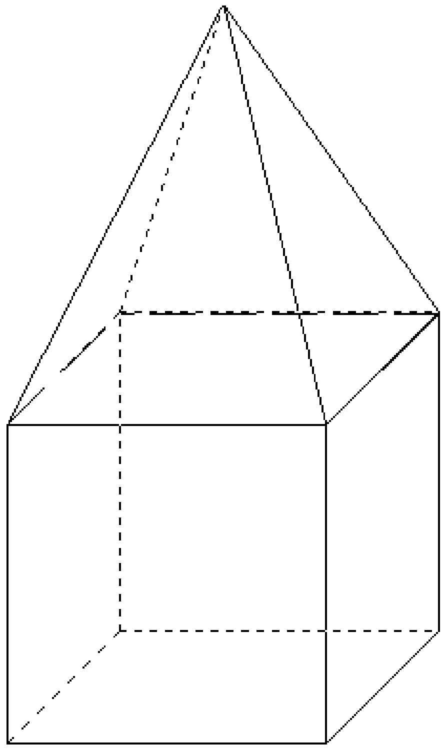


Figure 2.40.2 (Pyramid with cube)

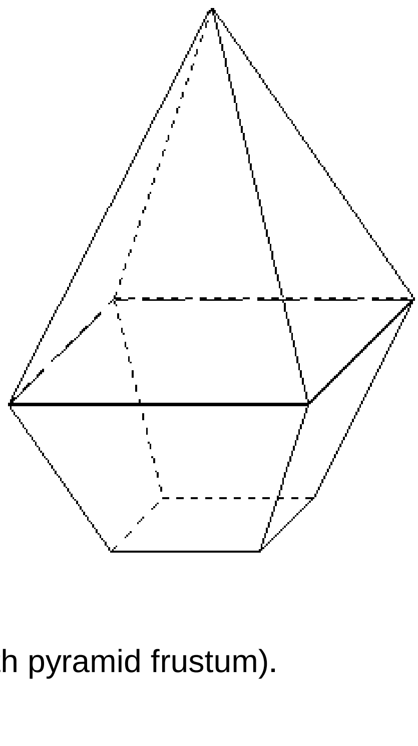


Figure 2.40.3 (Pyramid with pyramid frustum).

2.2.11 Inscribing and circumscribing pyramids

We select the "Inscribe" of square pyramid in a cube. We illustrate the result in a spatial corner diagram and in a three plane projection (Figure 2.41.1/2); after a rotation it looks like in figure 2.41.3.

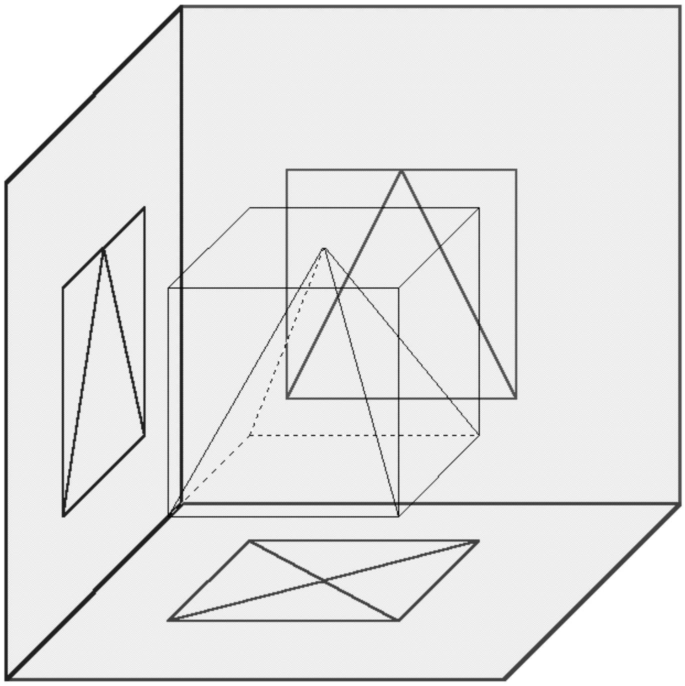


Figure 2.41.1

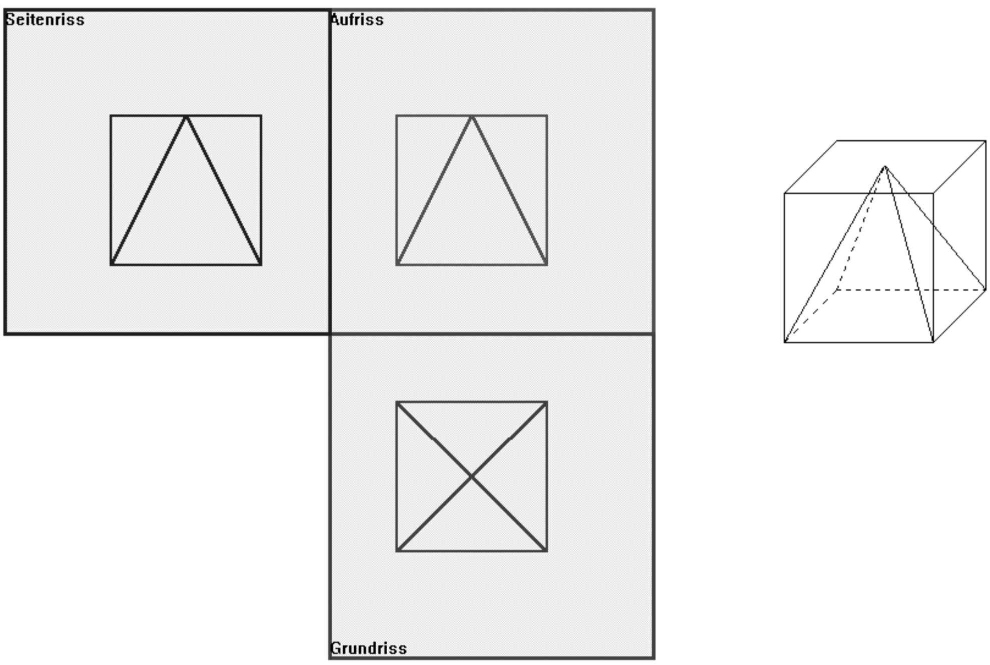


Figure 41.2

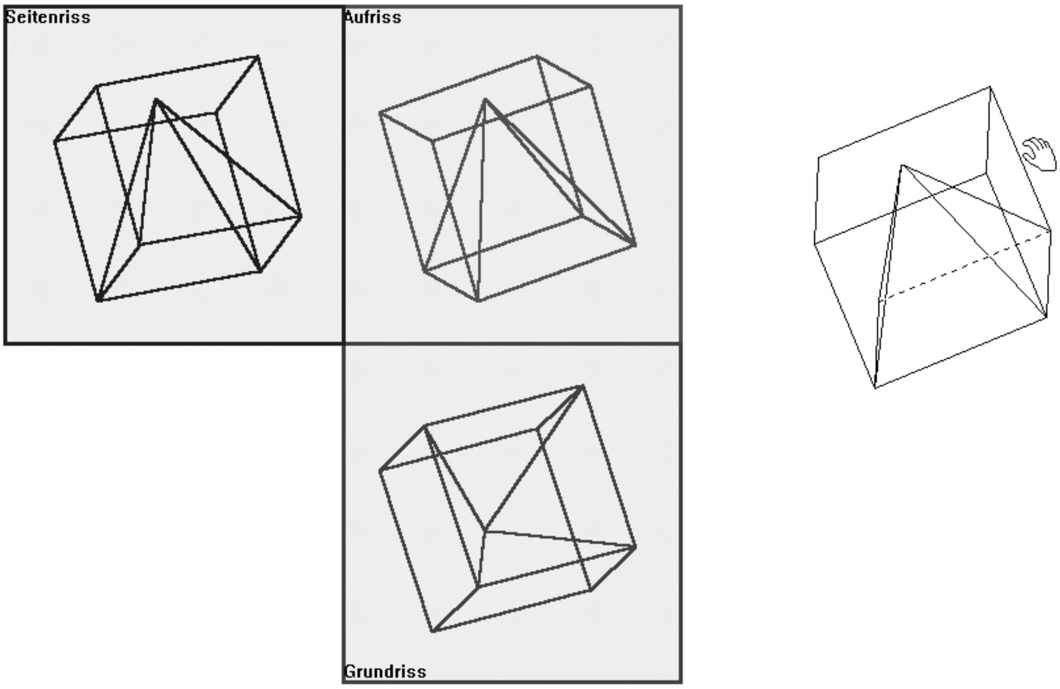


Figure 2.41.3

We inscribe a cube into a square pyramid or circumscribe a pyramid round a cube (Figure 2.42.1/2).

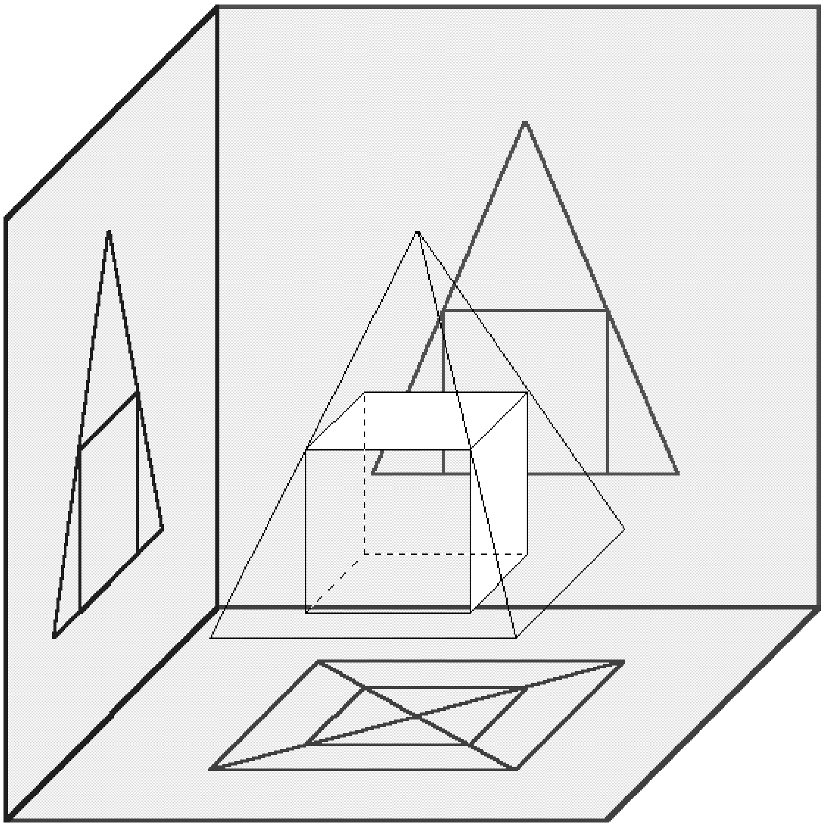


Figure 2.42.1

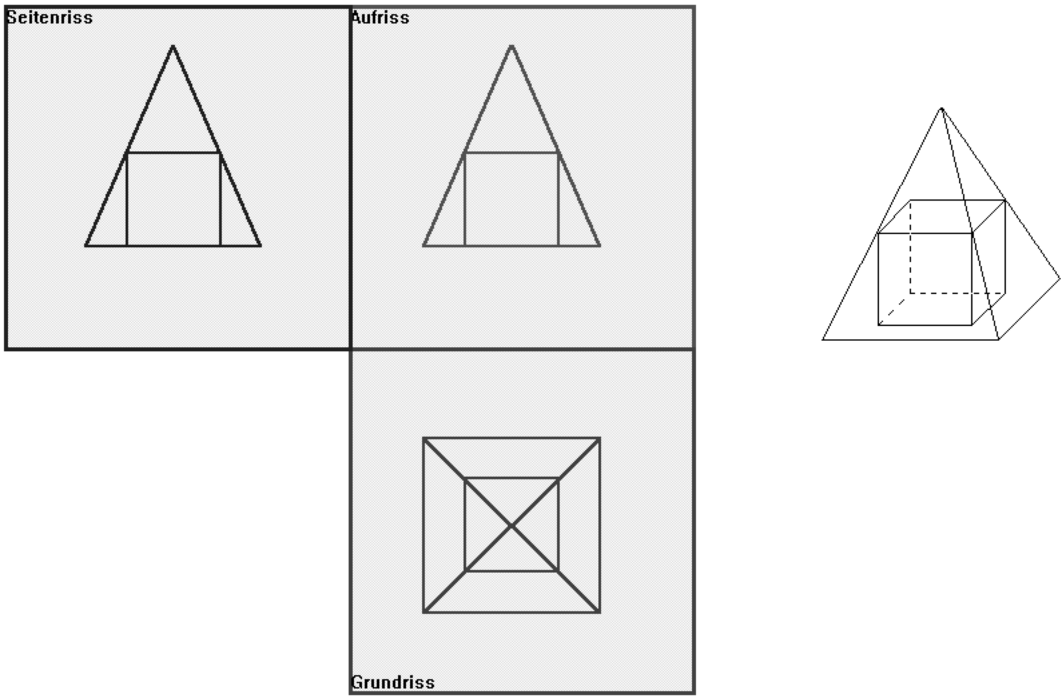


Figure 2.42.2

We also can inscribe a cylinder or sphere into a square pyramid (Figure 2.43/44).

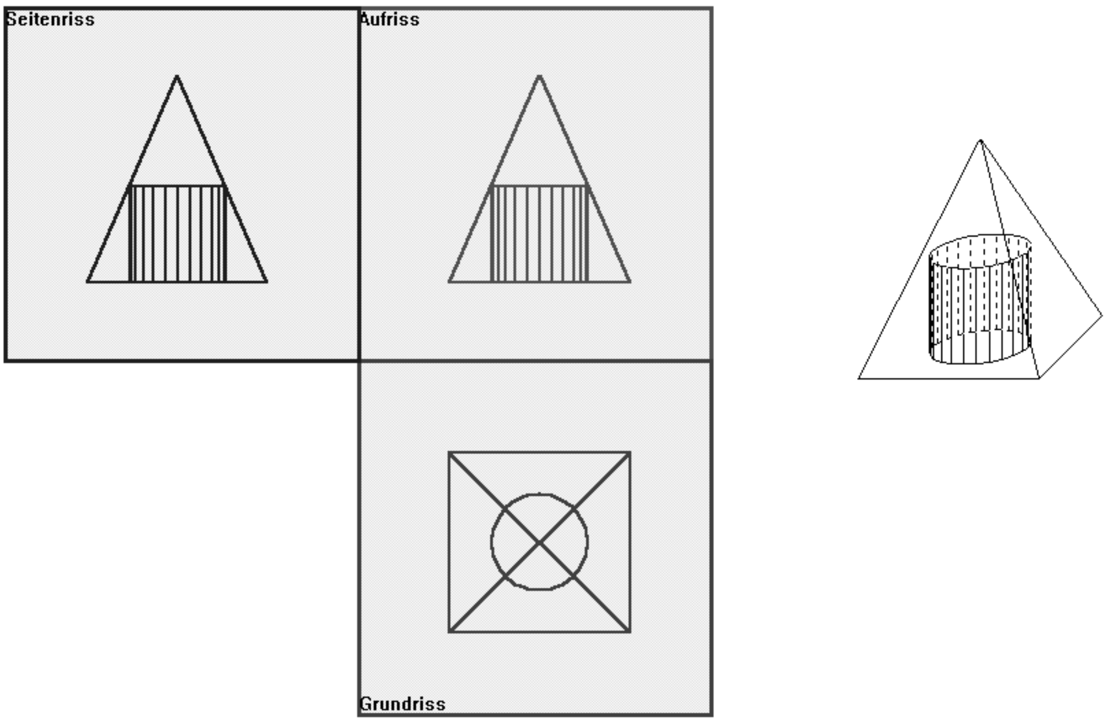


Figure 2.43

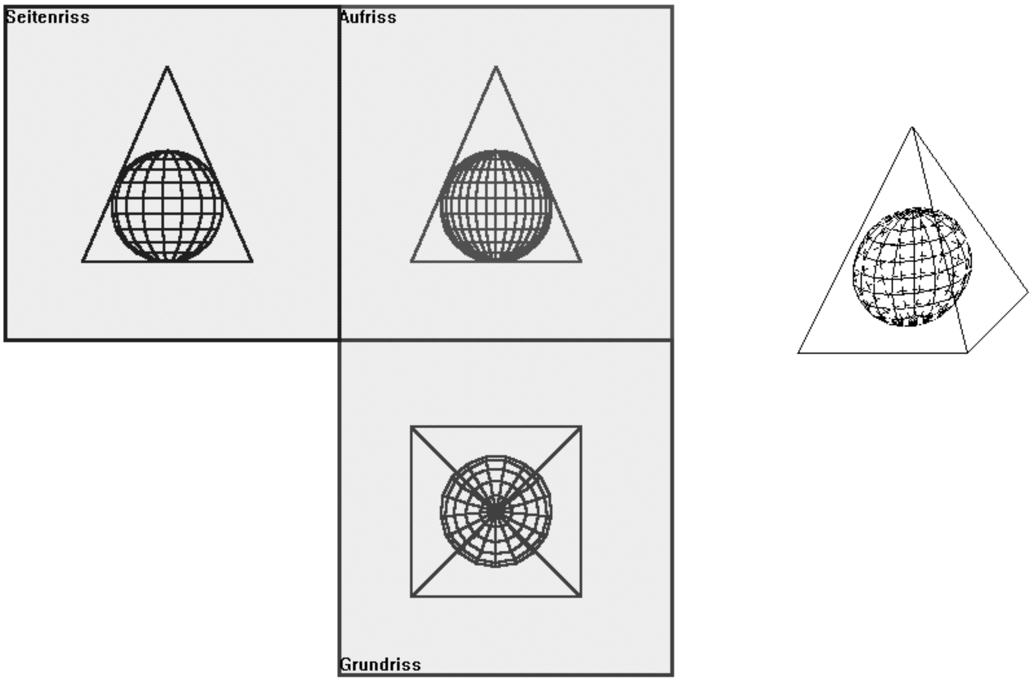


Figure 2.44

2.2.12 From the pyramid to the "cone"

Figures 2.45.1- 2.45.4 illustrate the approximate transition of the 4-sided pyramid -- to the "cone" of equal height and equal radius of the base area; the corresponding nets are also displayed, showing the regular base and the lateral surface of the pyramid for comparison purposes.

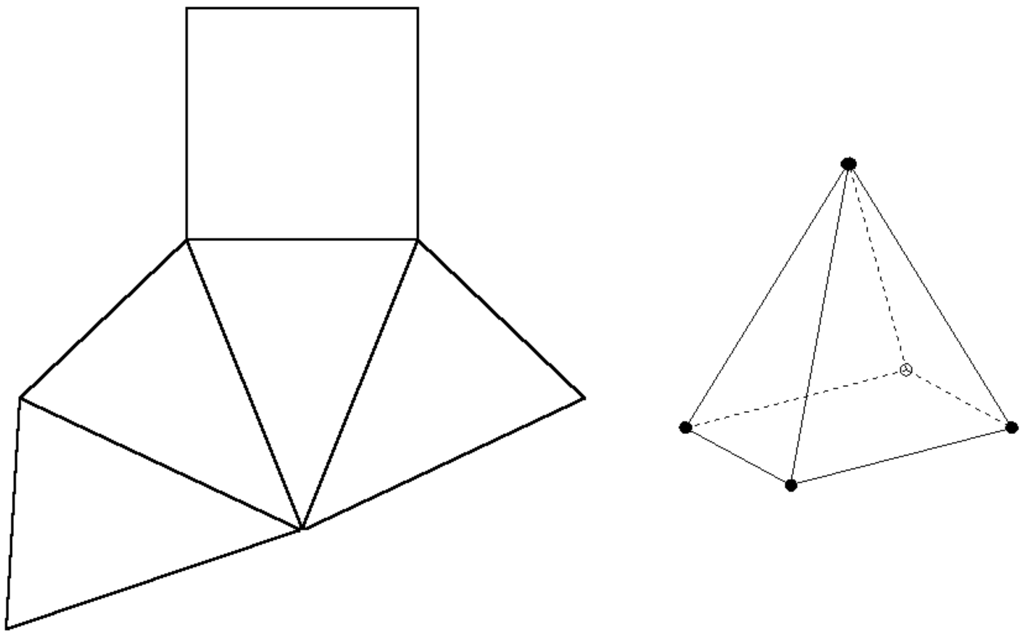


Figure 2.45.1

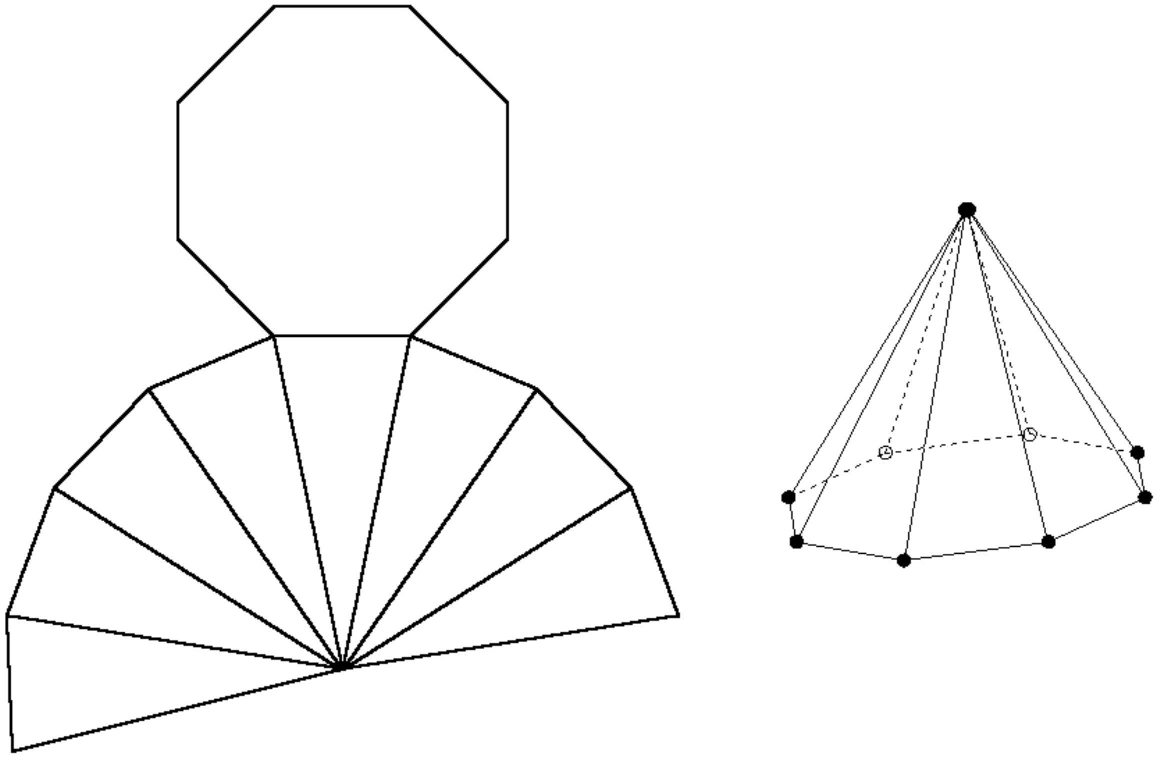


Figure 2.45.2

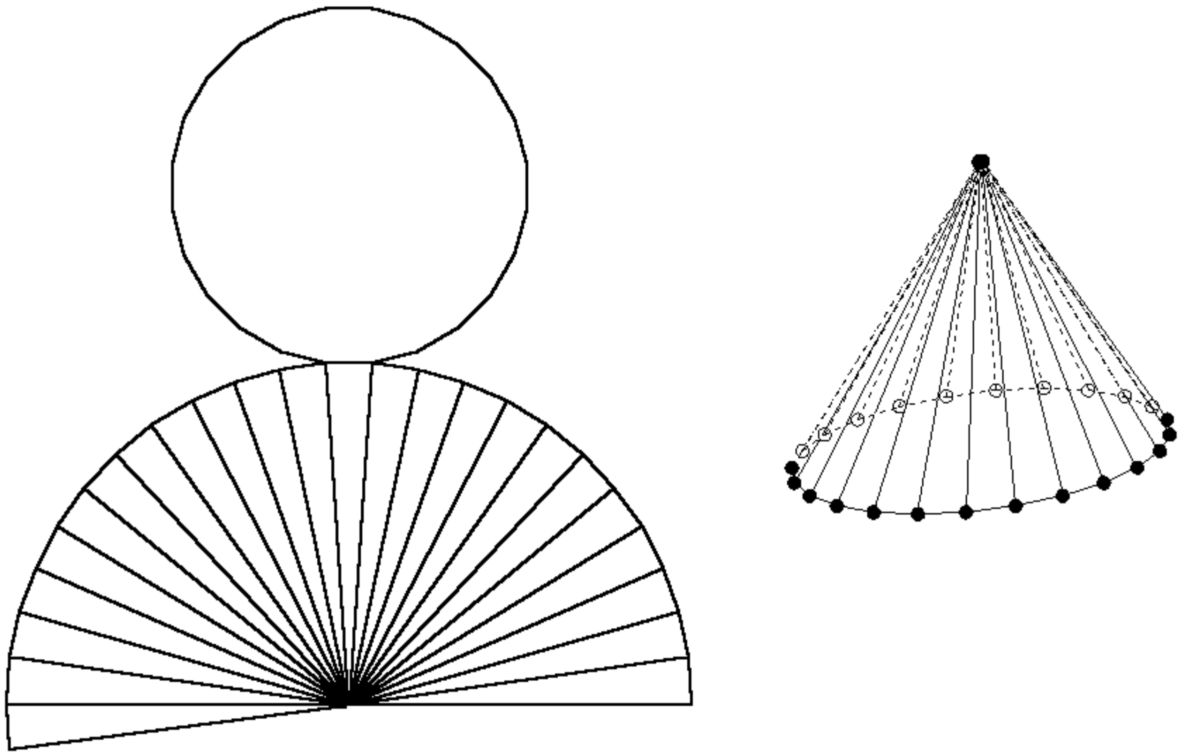


Figure 2.45.3

The solid in figure 2.45.3 can also be created with the cone button; however, we recognize by counting edges or vertices that it is actually.a 24-sided pyramid. In KOERPERGEOMETRIE a cone is always approximated by such a pyramid. –By dragging the top of the cone to a skew we get an aestheticaly looking net (Figure 2.46).

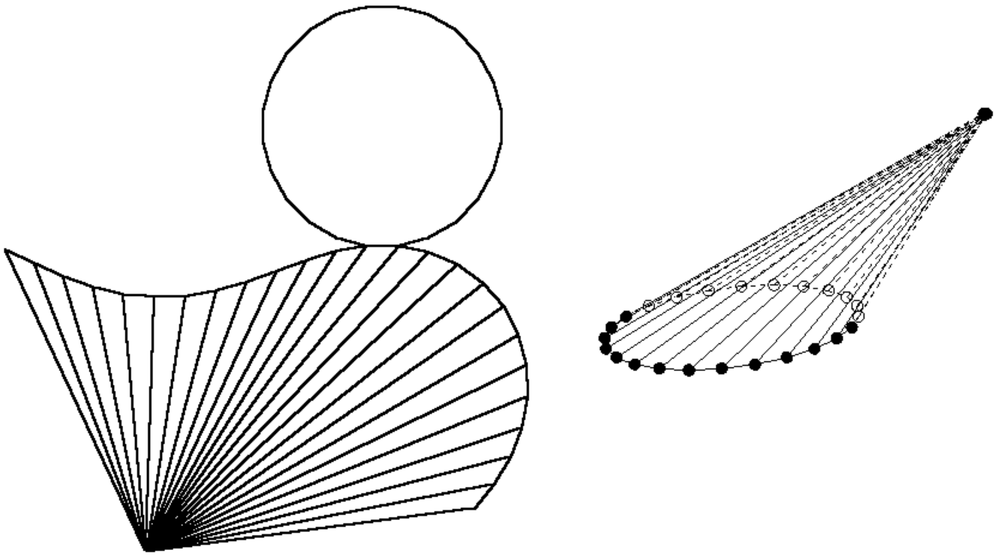


Figure 2.46

The cone can be created also as rotation solid. To this the triangular rotation profile must correspondingly be adjusted by dragging (Figure 2.47.1).

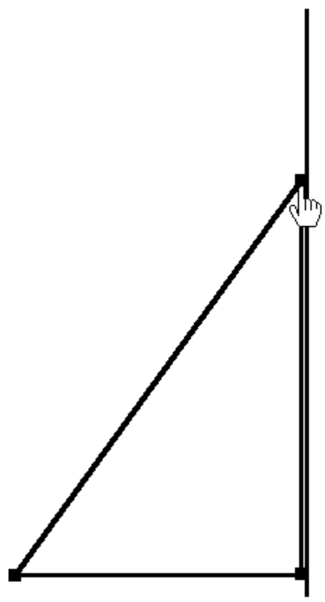


Figure 2.47.1

Depending on the selected number n of facets one gets a n-sided pyramid (Figure 2.47.2, n = 4: square pyramid);

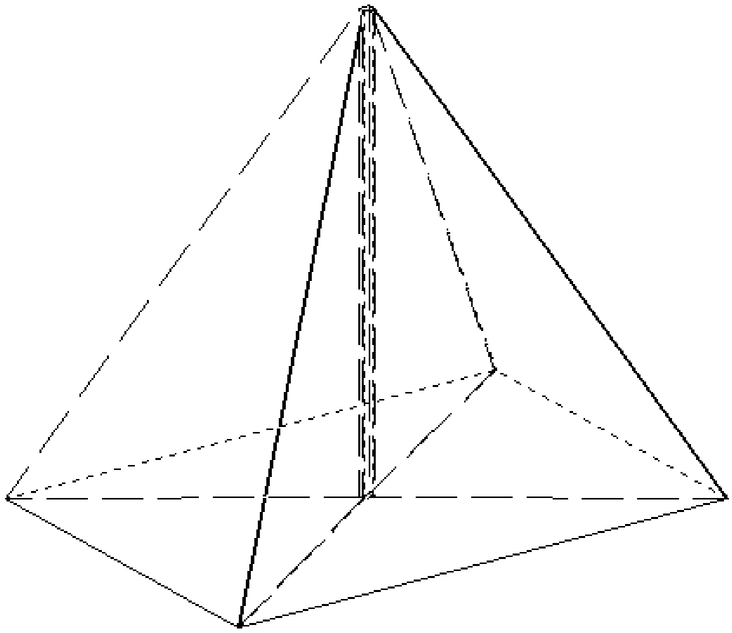


Figure 2.47.2

for the maximum number n=24 for facets we approximately get a cone (Figure 2.47.3).

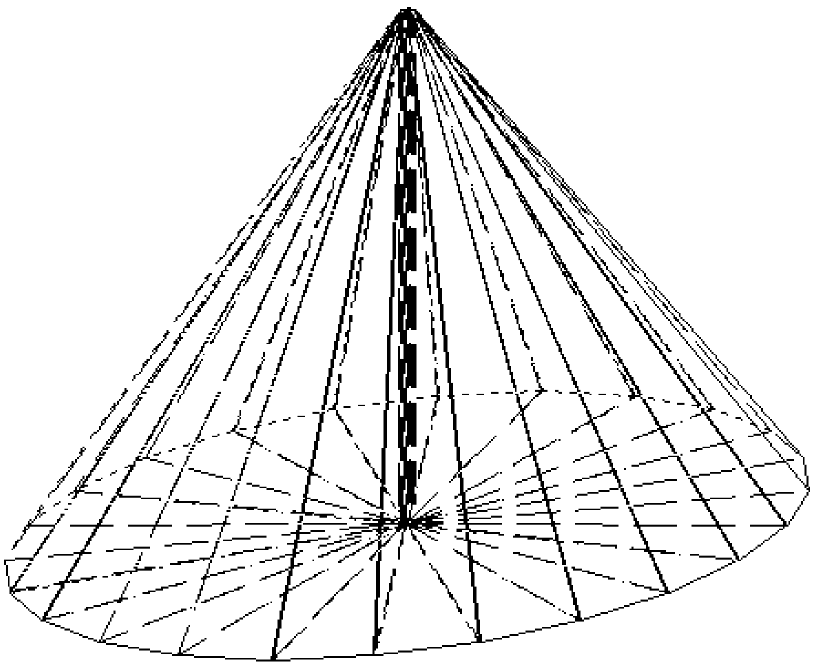


Figure 2.47.3

By dragging the rotation profile the production of the rotation solid can be demonstrated (Figure 2.47.4).

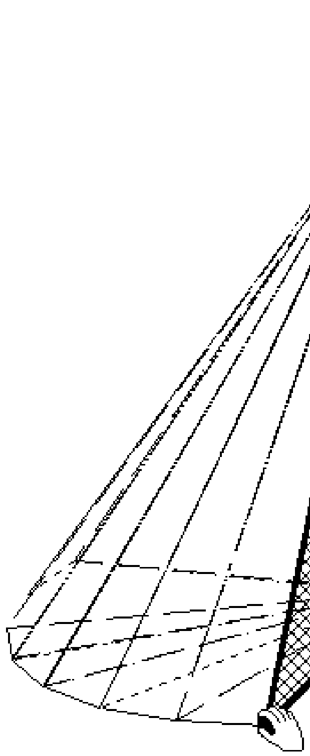


Figure 2.47.4

2.2.13 Different kinds of conical section

A plane parallel to the base cuts the cone to form a circle (Figure 2.48.1).

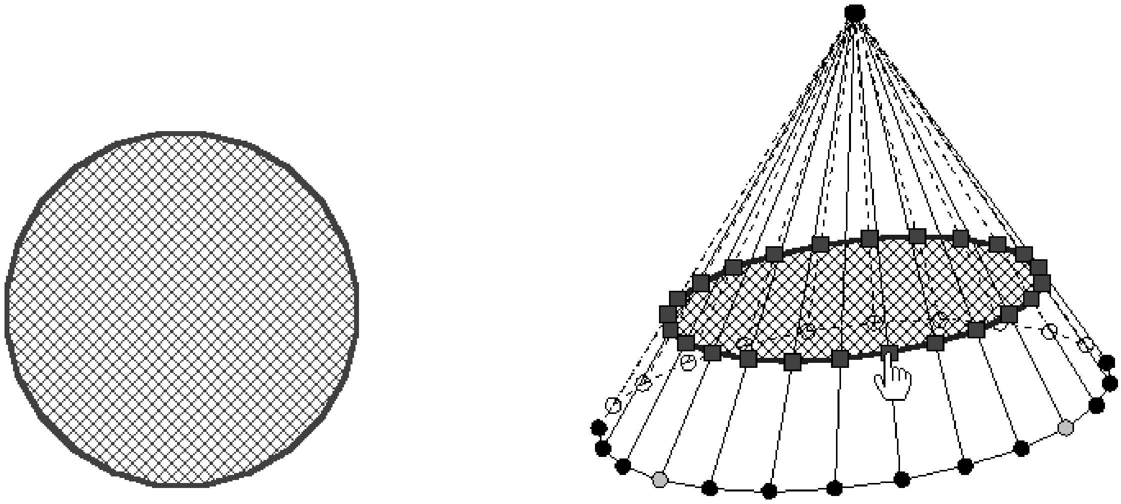


Figure 2.48.1

The type of conical section depends on the the size of the angle compared with the angle between the cone generating line and the base of the cone: we get an ellipse if this size is smaller (Figure 2.48.2); a hyperbola (Figure 2.48.3) if this size is greater and a parabola if this size is equal.

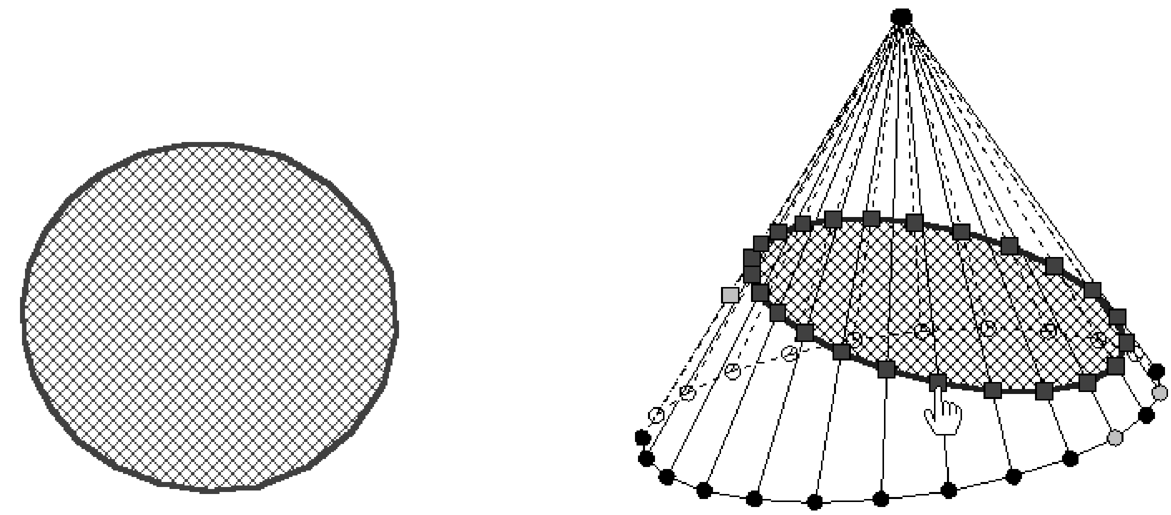


Figure 2.48.2

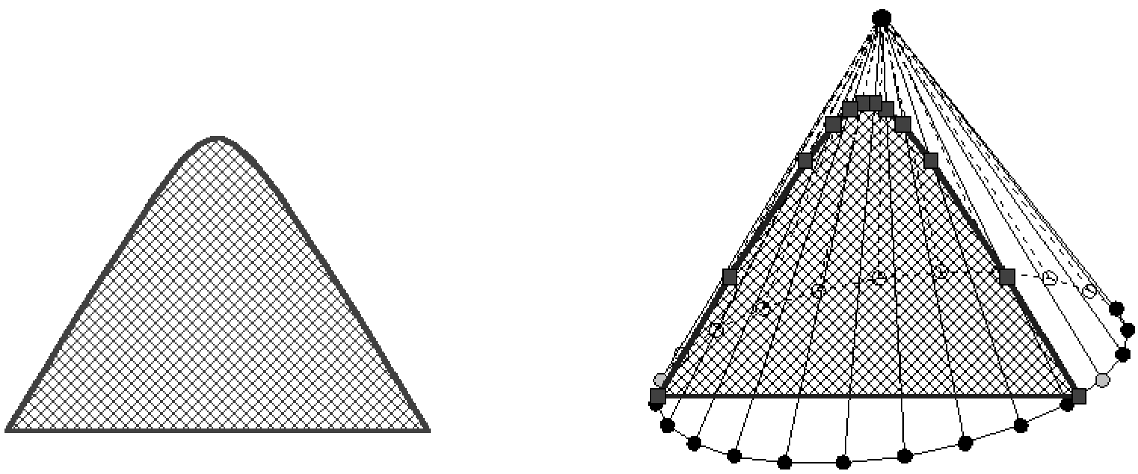


Figure 2.48.3

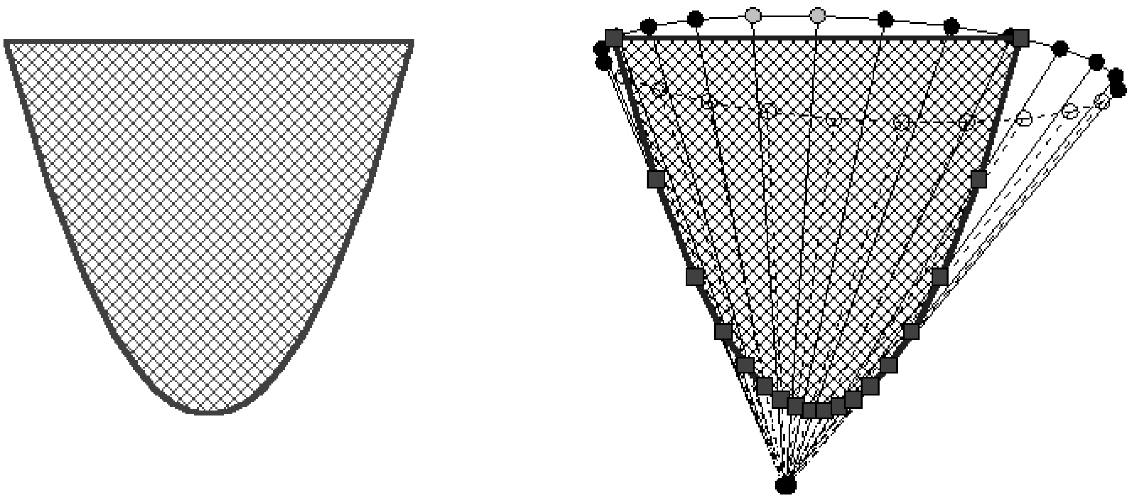


Figure 2.48.4

With the corresponding tools the angles in question can be measured.
We now execute all the cuts of the sections mentioned and represent the nets of the section solids in question: Cone frustum with circular base (Figure 2.49.1),

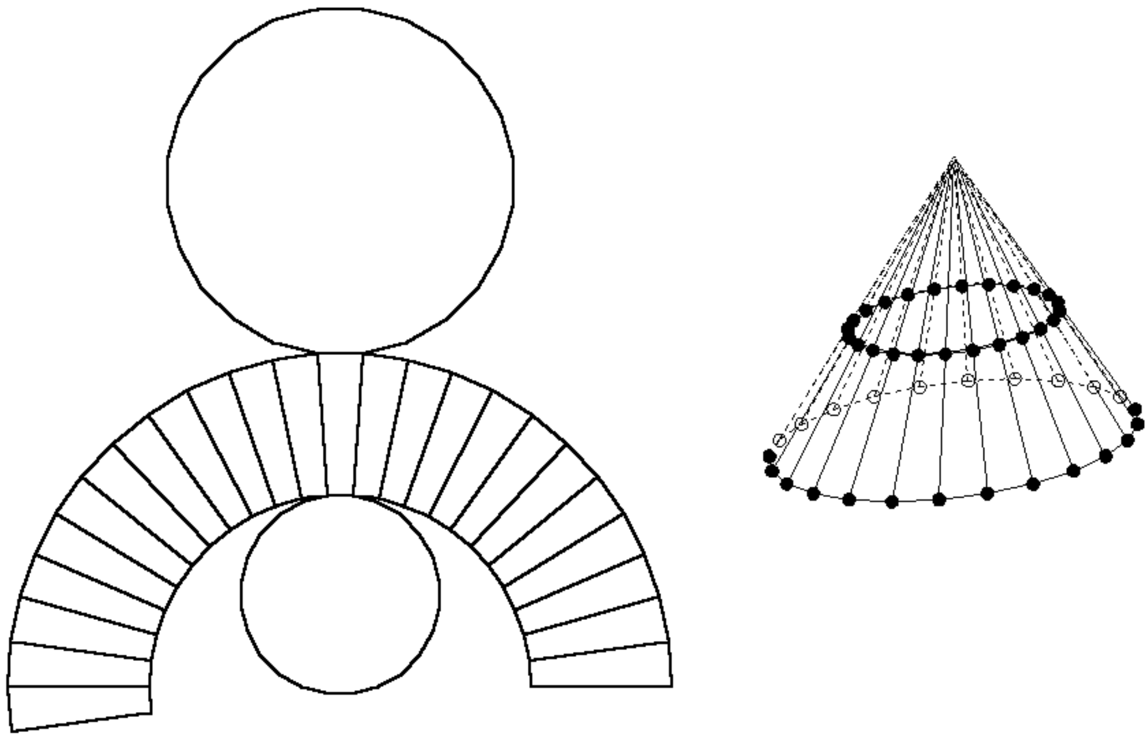


Figure 2.49.1

Cone frustum with an elliptic top (Figure 2.49.2a)

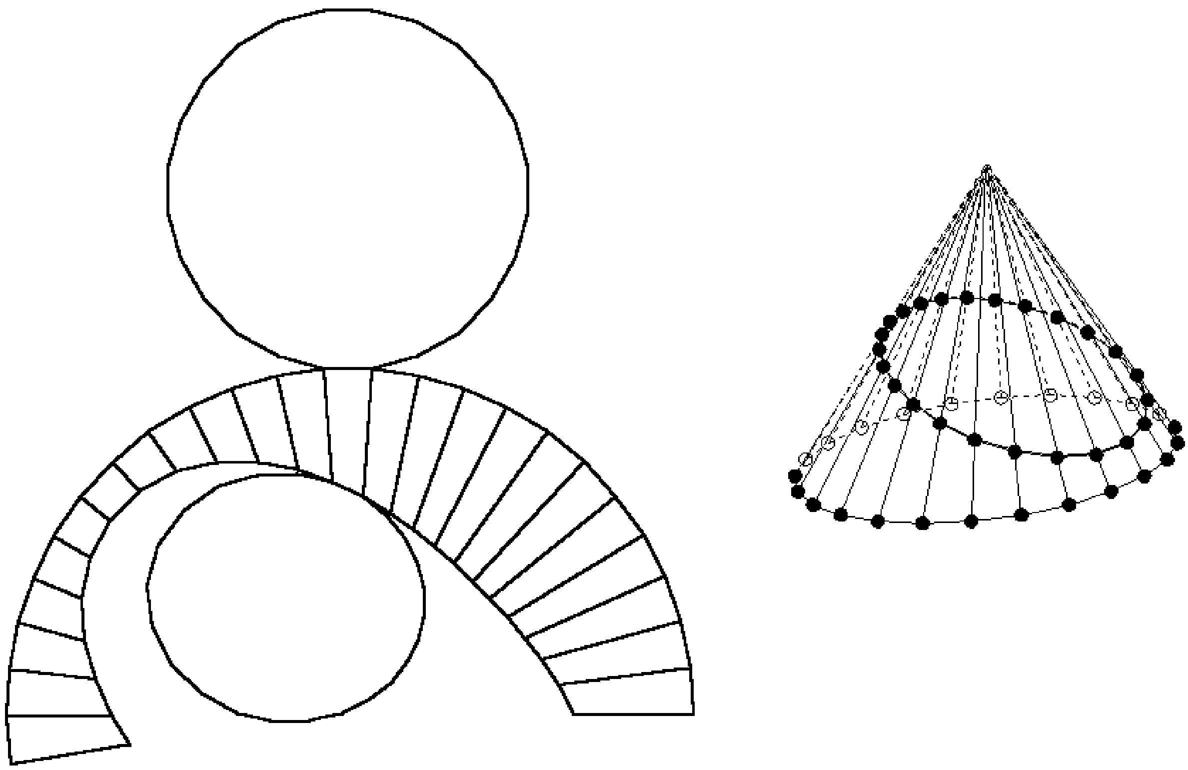


Figure 2.49.2a

and cone with elliptic base (Figure 2.49.2b),

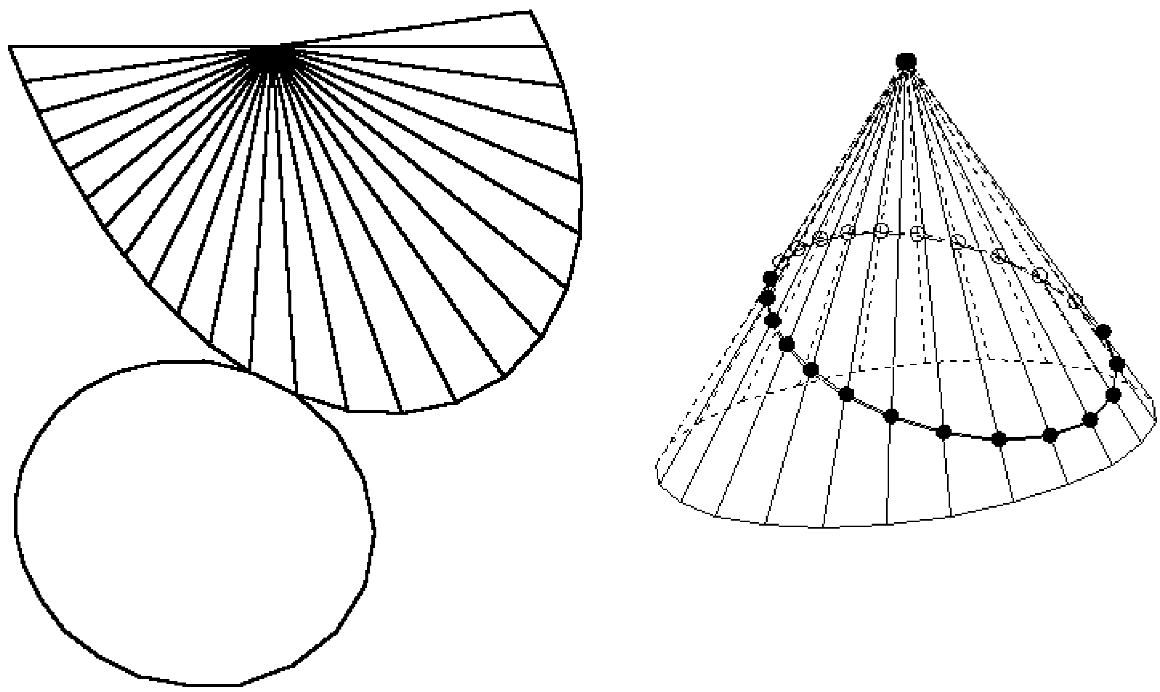


Figure 2.49.2b

Subsolids of the cone with sections of hyperbolic shape (Figure 2.49.3a/b).

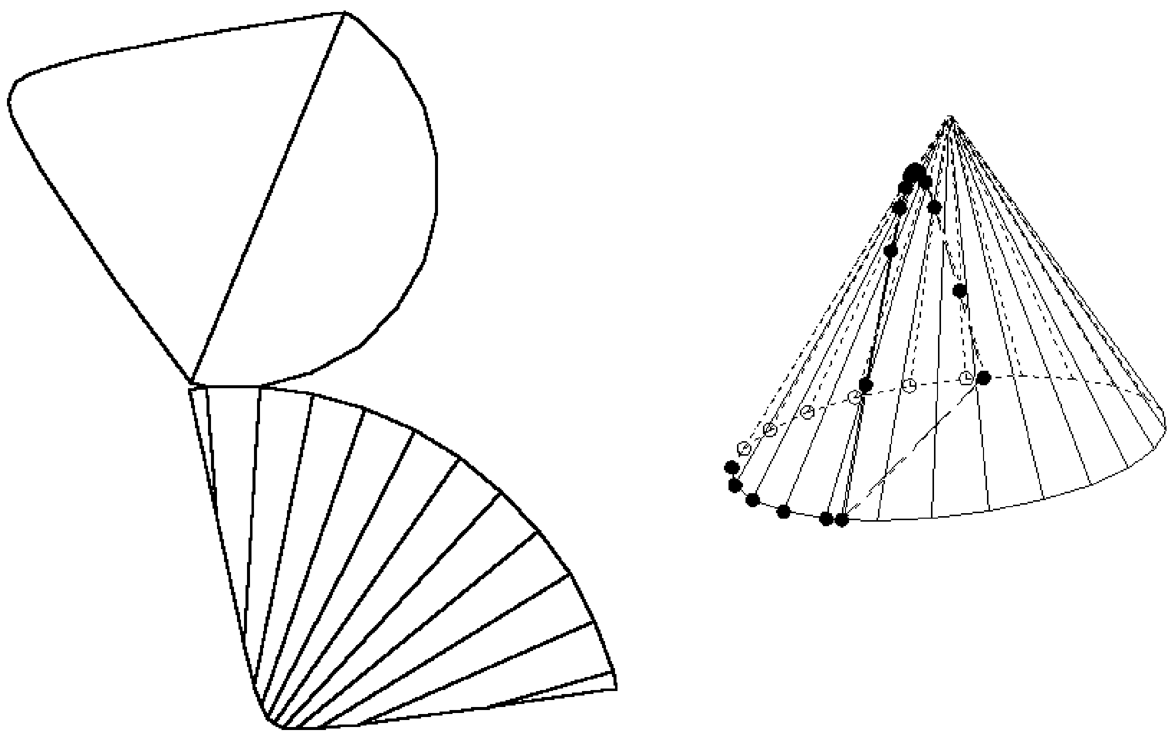


Figure 2.49.3a

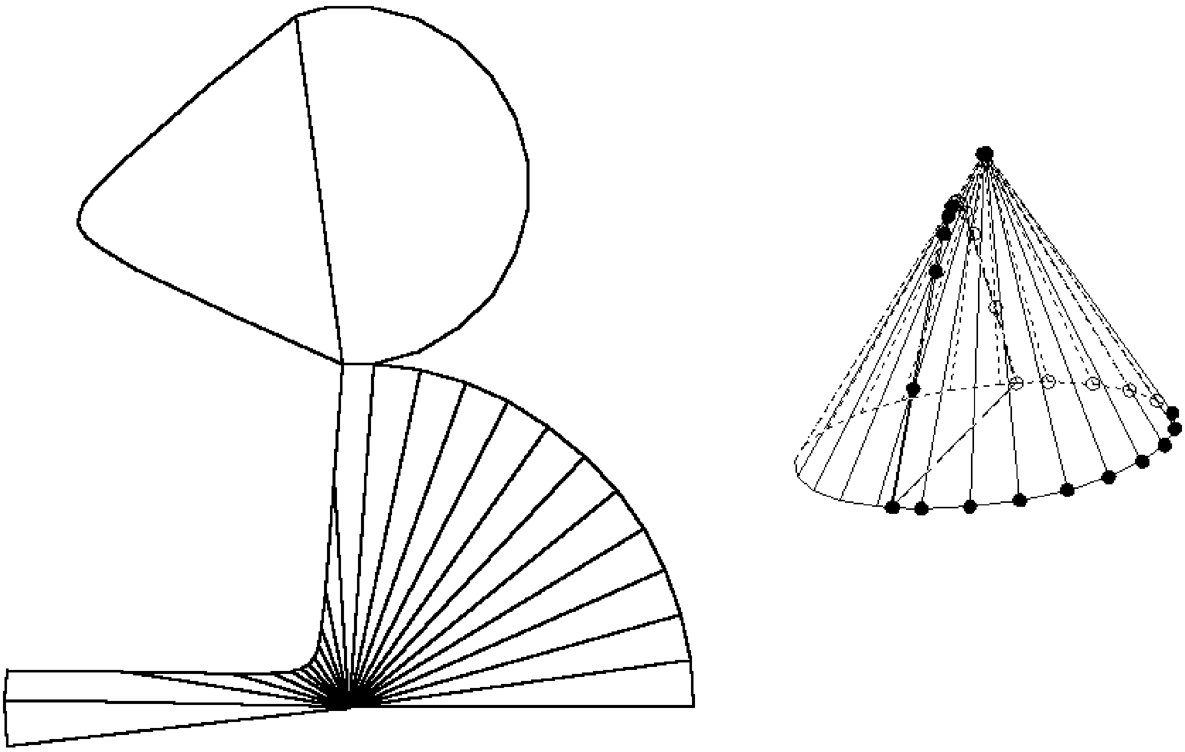


Figure 2.49.3b

Subsolids of the cone with sections of parabolic shape (Figure 2.49.4a/b).

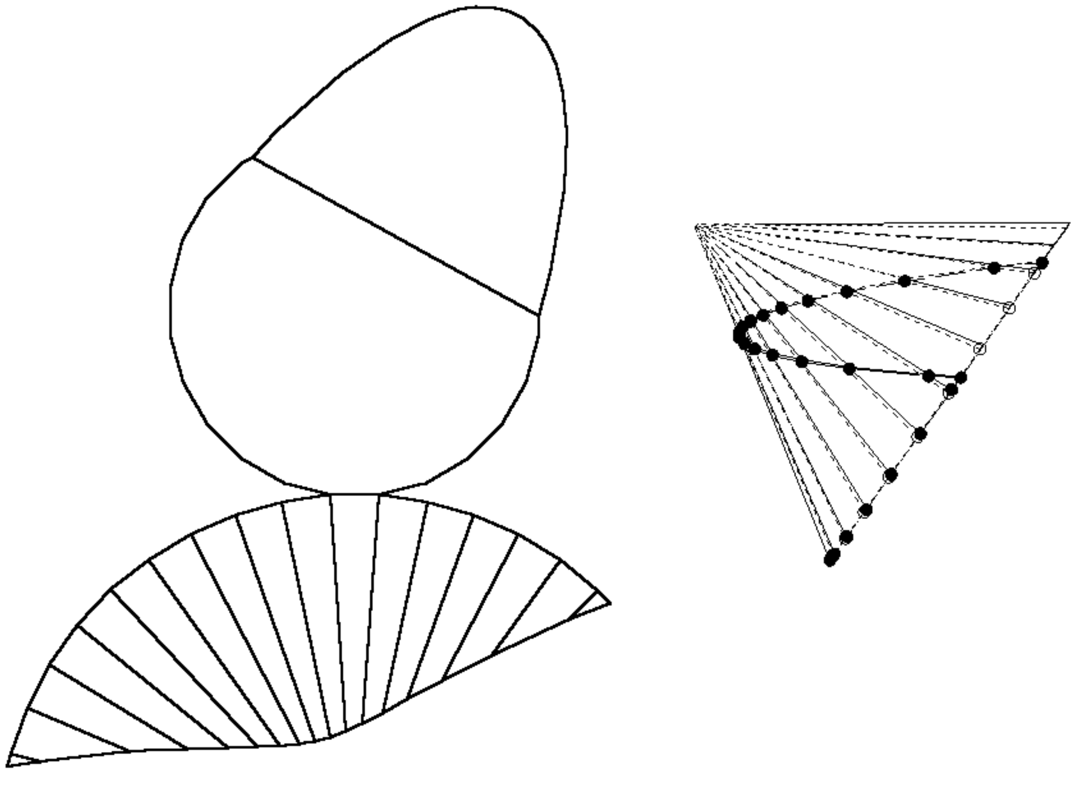


Figure 2.49.4a

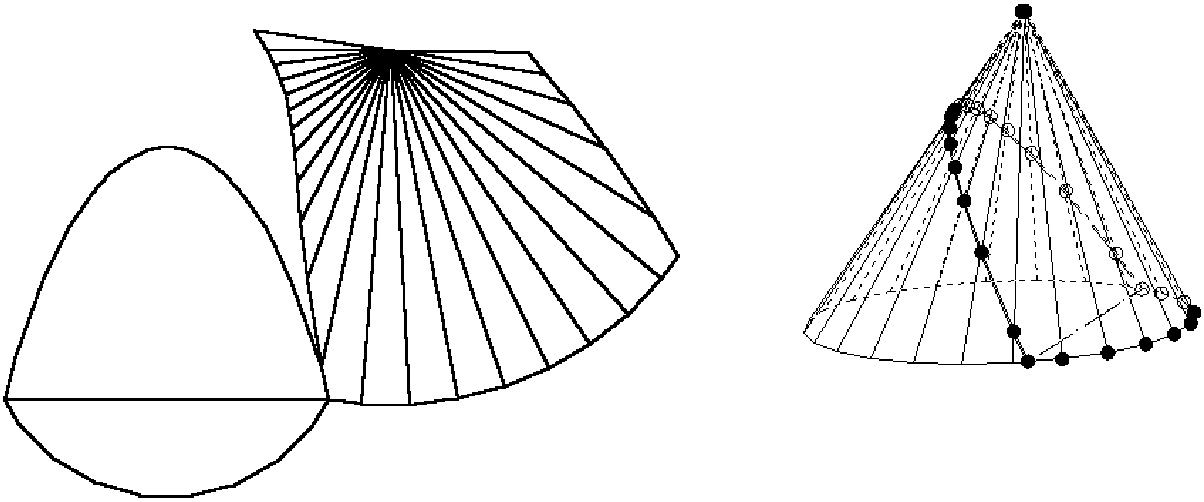


Figure 2.49.4b

The section outline can be anticipated before the execution, for example for the elliptic section (Figure 2.50).

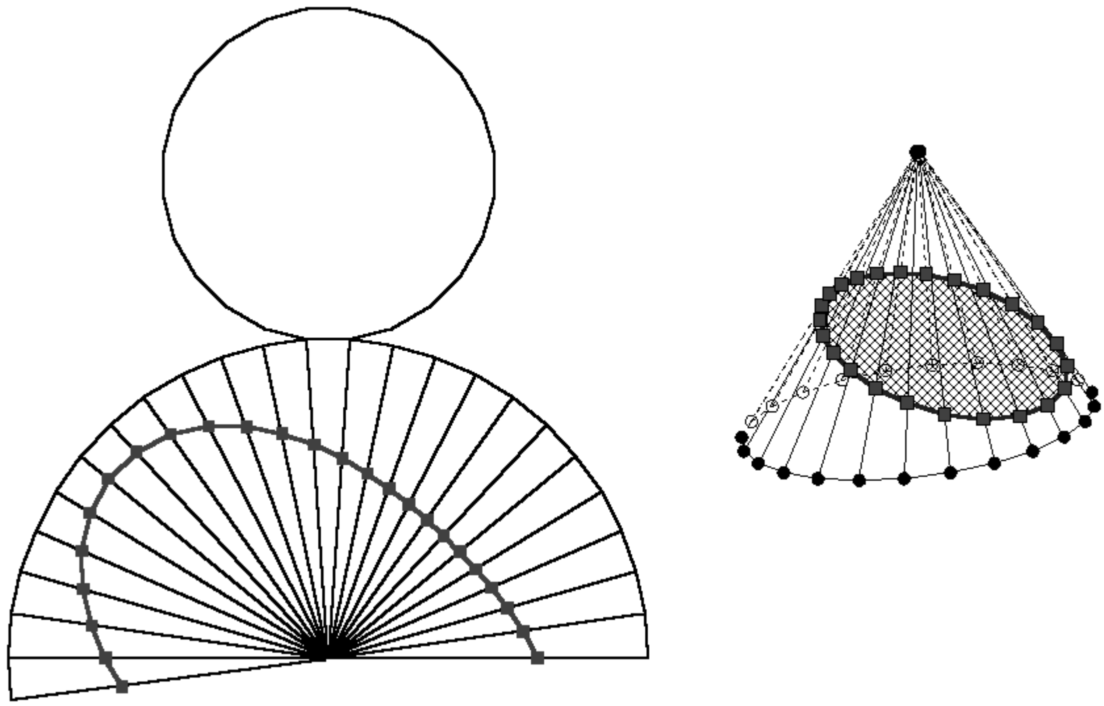


Figure 2.50

2.2.14 From cone to cylinder

The cylinder can be understood as a cone of which the top is dragged to „infinity" (Figure 2.51.1/2).

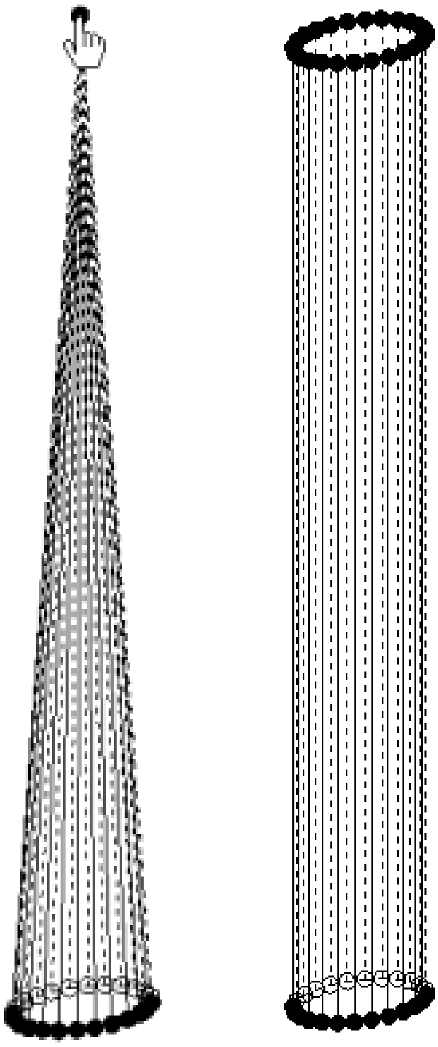


Figure 2.51.1/2

A net of the upright circular cylinder (Figure 2.52.1) is transformed dynamically to a net of an oblique circular cylinder with dynamic appearance (Figure 2.52.2).

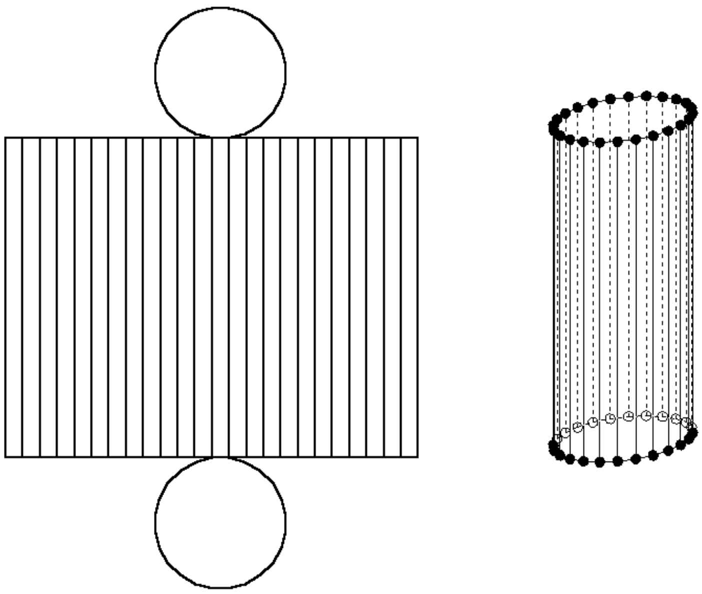


Figure 2.52.1

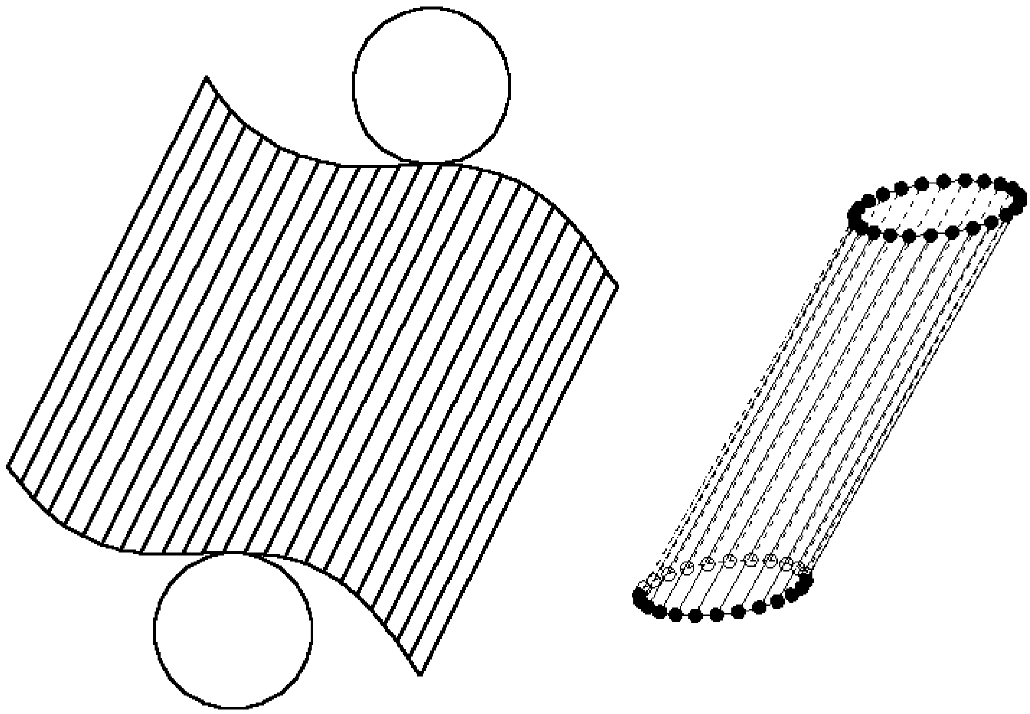


Figure 2.52.2

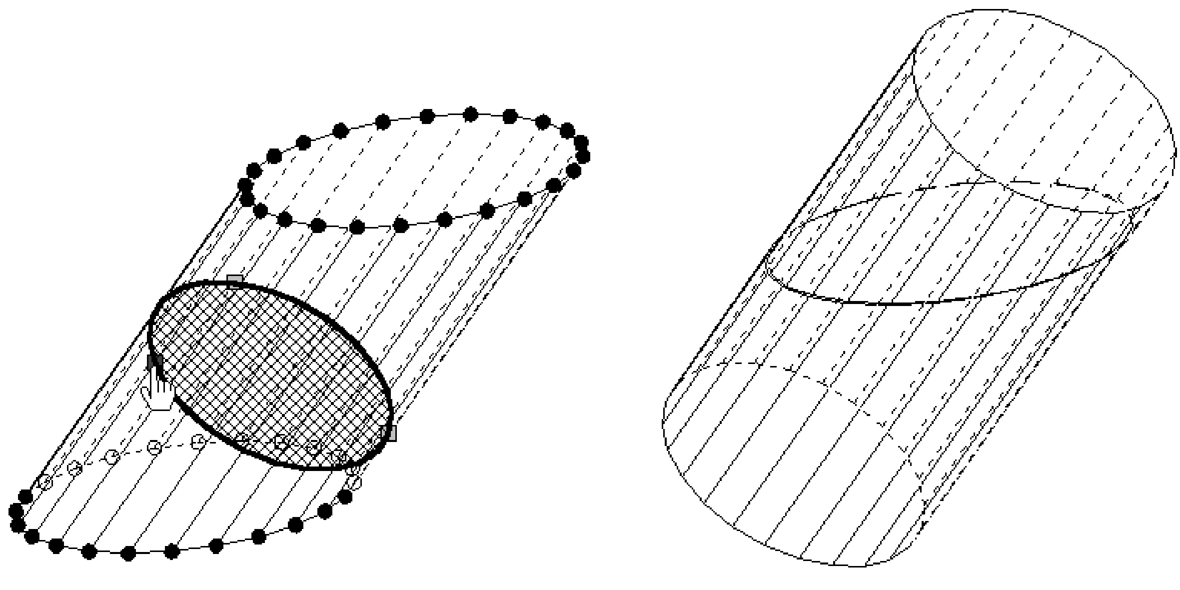


Figure 2.53.1/2

We make an oblique cylinder from an upright one, by executing a section vertically to a generating line and by suitably changing the position of one of the subsolids (Figure 2.53.1/2). The elliptic section produces a corresponding net like the skin of a sausage cut off (Figure 2.54.1/2).

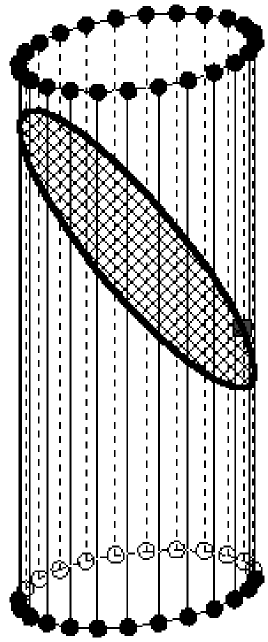


Figure 2.54.1

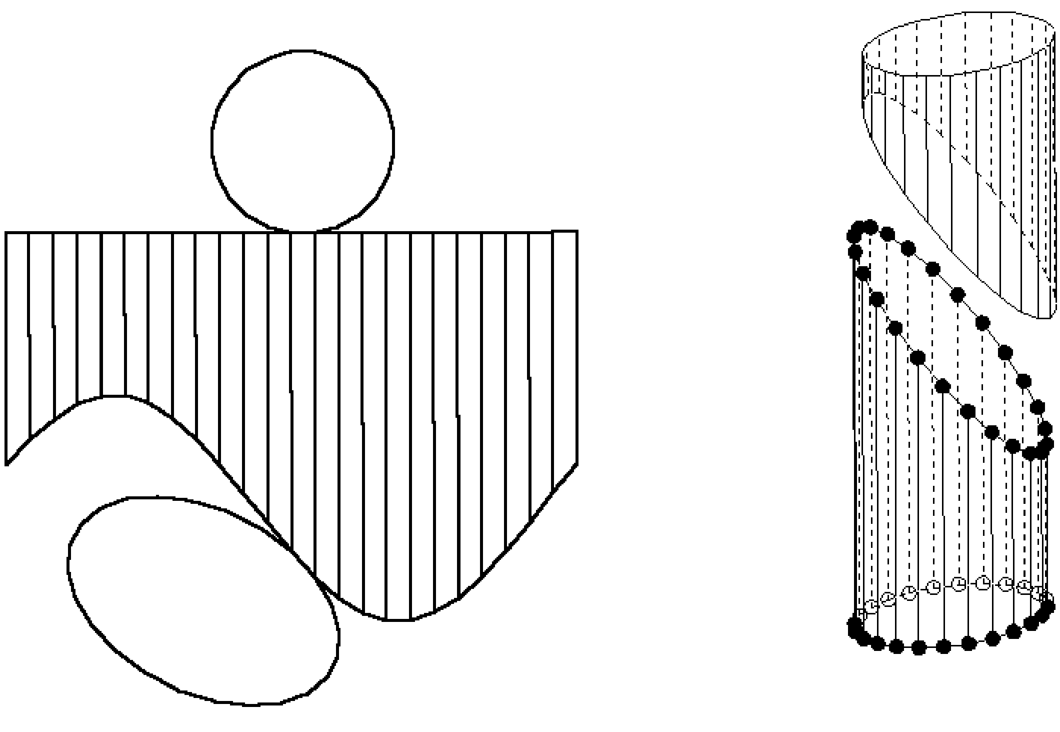


Figure 2.54.2

A creeper is climbing economically around a cylindrical tube, i.e. taking the shortest path (Figure 2.55); this is shown in the net where the path is seen to be a straight line segment ...

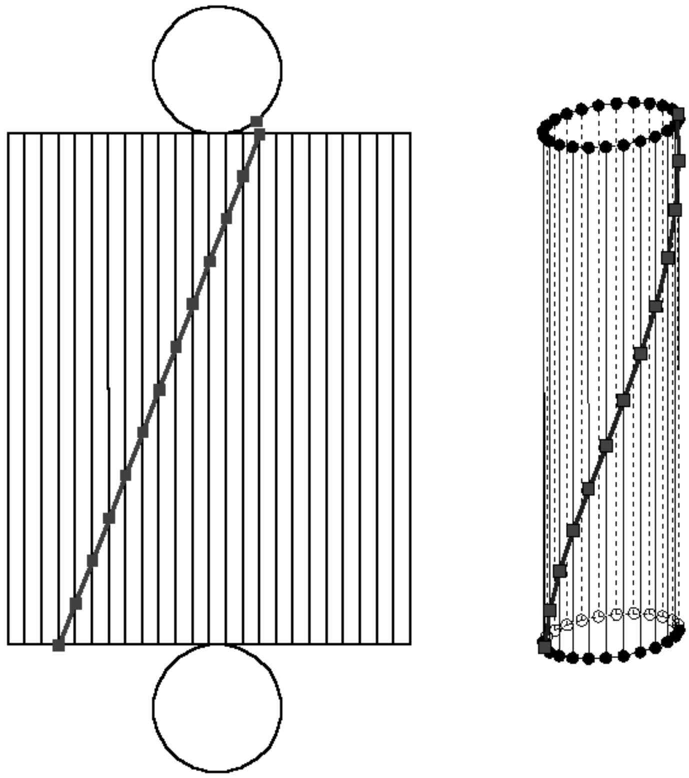


Figure 2.55

We conclude our excursion in KOERPERGEOMETRIE with the construction of a section solid (so-called multi-purpose stopper), which must fit to each of the three holes of the stencil given in figure 2.56.

Figure 2.56

We adapt a cylinder in that way that its height is equal in size to its diameter (Figure 2.57.1).

In order to adapt this solid to the profile of an equilateral triangle we insert two corresponding sections into the cylinder (Figure 2.57.2).

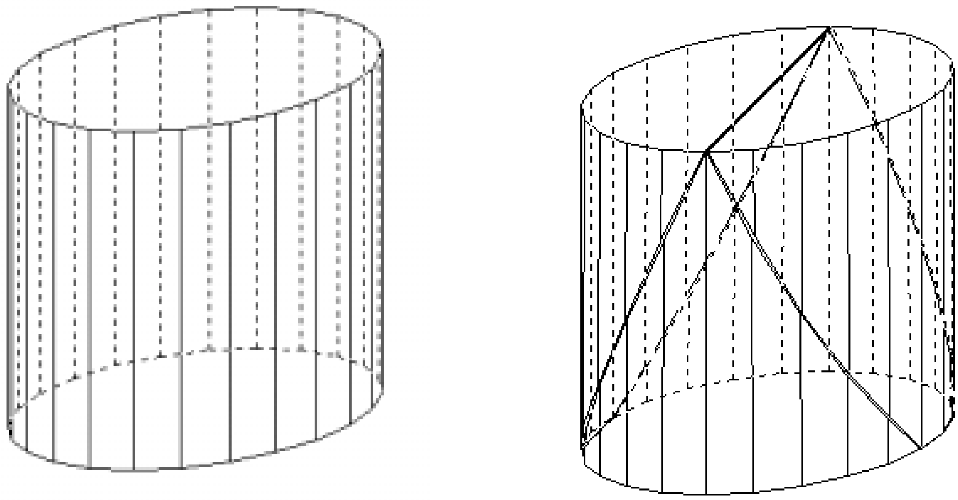


Figure 2.57.1/2

Figure 2.57.3 shows the separation of the sectional solids.

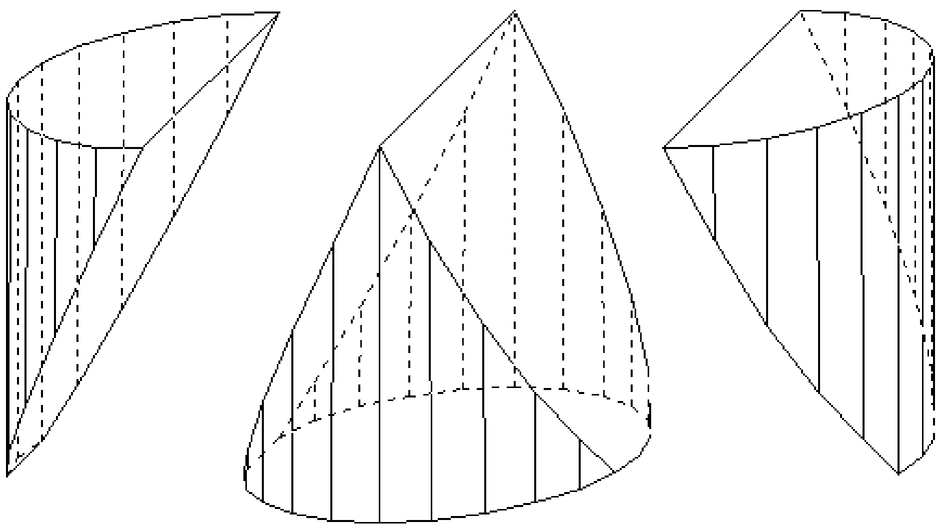


Figure 2.57.3

For tactile perception we make the multi-purpose stopper accessible to us with the generation of a net (Figure 2.57.4).

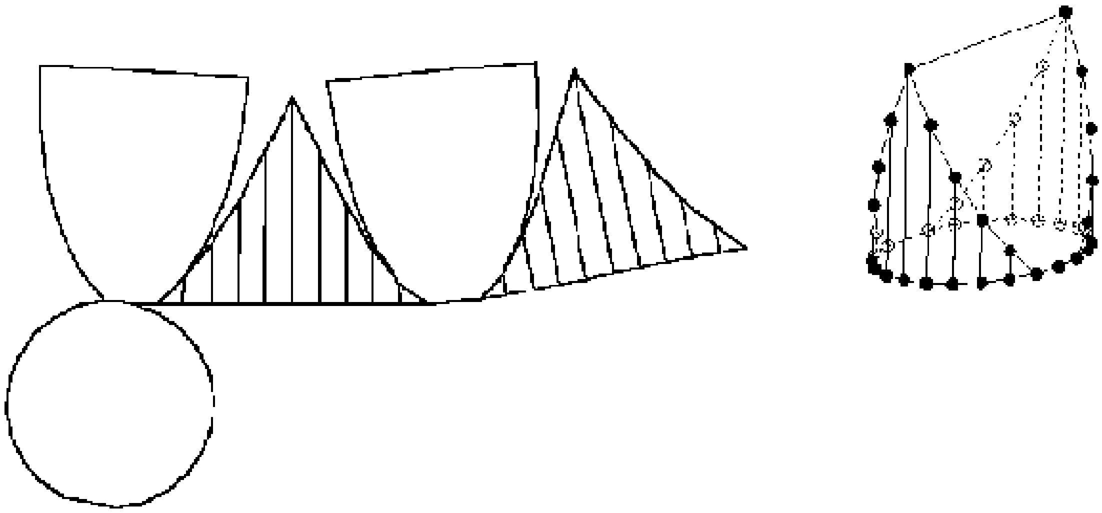


Figure 2.57.4

We can recognize in the three plane projection that the constructed solid has the desired qualities (Figure 2.57.5).

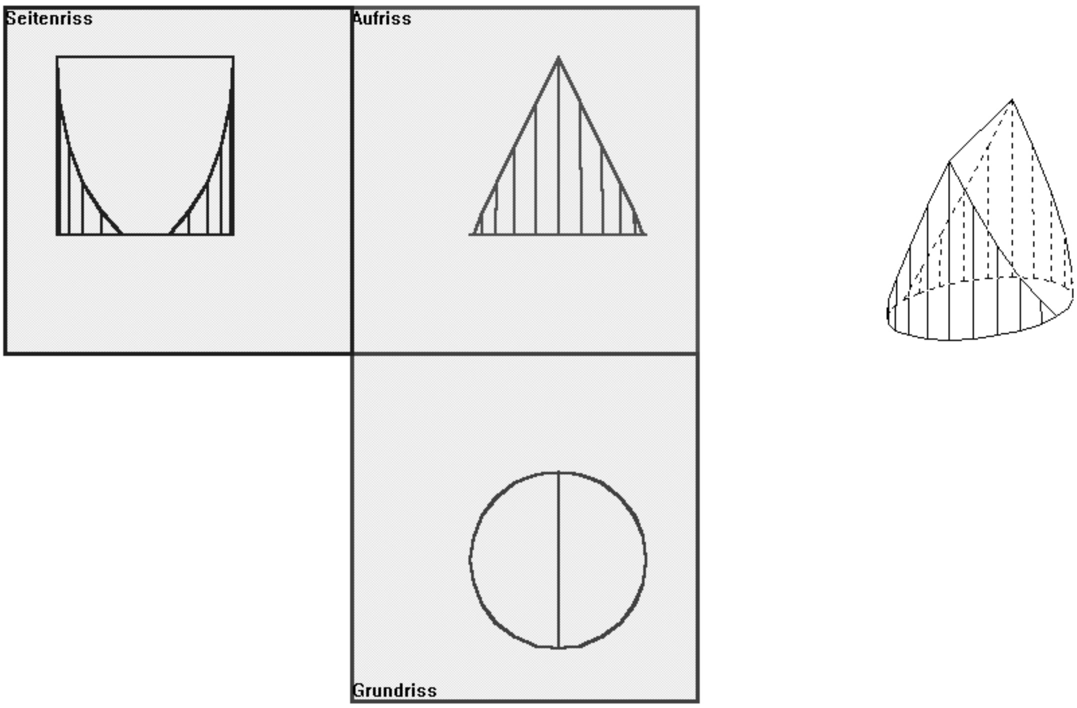


Figure 2.57.5

2.3 COMPUTER SUPPORTED TREATMENT OF (SPATIAL) GEOMETRIC CALCULATIONS

We explain new standards for solving of geometrical calculation tasks depending upon suitable choice of computer tools. We can distinguish the following essential kinds of computer supported solution:

- **computer graphical solution**
- **computer numeric and algebraic solution.**

Computer graphical solution depends on the figural representation of the solid to be calculated.in a graphical tool like KOERPERGEOMETRRIE with its possibilities of varying by dragging, constructing and measuring. Of course a graphical solution is possible only for specific calculation tasks, i.e. tasks without algebraic treatment of variables. The advantage of solving consists in the experimental, clear geometric understanding of the task without hindrance by arithmetical and algebraic barriers.

Computer numeric and algebraic solving means the solution of specific and general geometric tasks with support of computer numeric and algebraic components for mathematical support programmes, Those used here beeing DERIVE and MATHEMATICA.

Two kinds of these solving methods are to be distinguished:

- the **"Simulation Method"** by means of DERIVE
- the **"Formulation Method"** (Ansatz-method) by means of MATHEMATICA.

The Simulation Method: The simulation method involves the process of manual solution using the support of automatic algebraic execution provided by the appropriate options. It entails building an "essential" equation (formula etc.) in which all variables with the exception of the sought-after variables are to be substituted. To obtain expressions for the variables to be replaced, one needs auxiliary equations, which can be resolved automatically. The substitutions are carried out interactively by means of the option "Substitute" or by definiting the global variables and by "Simplify". The essential equation is then resolved for the sought-after variable. The general calculation task is solved in that way. The given numeric data are to be inserted into the algebraic solution for the solution of a special calculation task in order to get an approximate or an exact numeric solution. One deduces the conditions for solvability from the algebraic solution. For further calculation and modular work one defines algebraically calculation macros with appropriate depend variables.

The Formulation Method:

A formulation consists of three sets:

–A set of linear and/or non-linear equations with target variables as well as auxiliary variables which together form a system of implicit or explicit algebraic equations.
–The subset of target variables for which the system of equations is to be solved.
–The subset of auxiliary variables which are to be eliminated.

For specific calculation tasks the concrete value assignments have to be added.

To find such a formulation, a knowledge of facts and procedures related to the problem is required, in particular appropriate heuristic knowledge. Some general instructions for the solving of a mathematical problem were given by G. Polya, 'father of mathematical heuristics in schools', in his well known book *How to Solve It?* (Princton, 1944). These instructions also contain the development of a solution plan in the narrow sense, the development of the above formulation.

HOW DOES ONE LOOK FOR THE SOLUTION?
FIRST –You must understand the problem
SECOND – Look for the connection between the data and the unknown.
You might have to look at auxiliary problems if an immediate connection cannot be found.
Then you must formulate a solution plan.
THIRD – Carry out your plan.
FOURTH – Check the solution you obtain.

Only the third step is essentially altered when an Auto-Solver is used: rather than 'Carry out your plan', it becomes 'Use an Auto-Solver to carry out your plan'.

Illustration of the application of the oulined methods

Statement of the problem:
Given: A square-based right pyramid has a volume 50 cm^3 and a surface area 100 cm^2.

To find: Determine the base length and height of the pyramid.

Graphical solution:
We select from the basic solid repertoire of KOERPERGEOMETRIE a right square-based pyramid with variable base length *a* and variable height *h*. We use the online-measurement and drag the pyramid until approximate values for *V* (= 50 cm^3) and *S* (= 100 cm^2) are achieved and then read off the corresponding base length (*a*) and height (*h*), obtaining *a* = 6.2 cm, *h* = 3.9 cm (Figure 3.1).

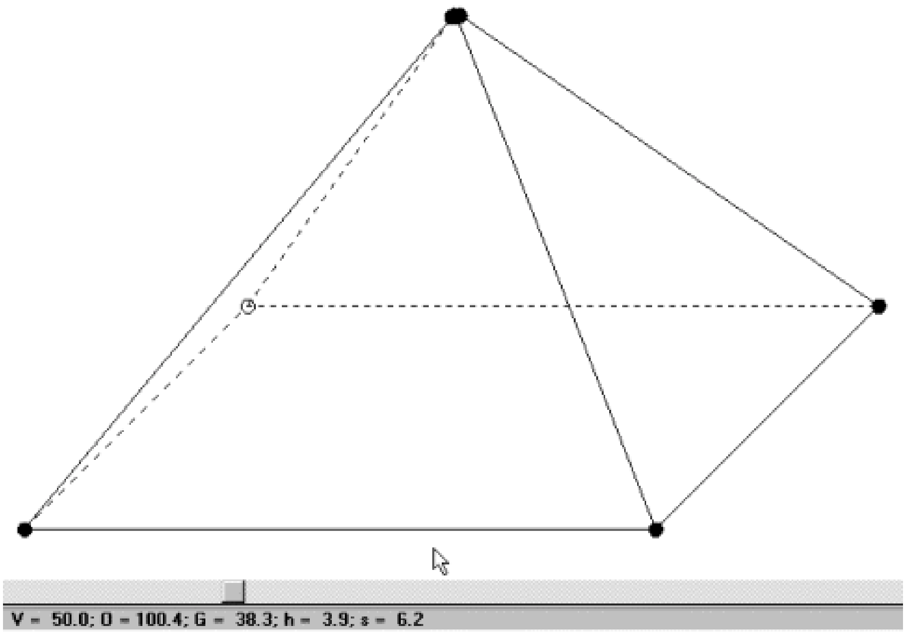


Figure 3.1

Solution by the Formulation-Method:

Numeric solution:

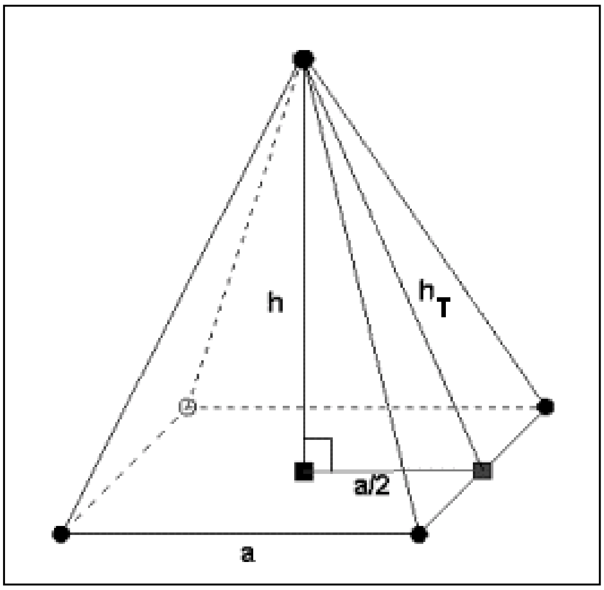


Figure 3.2

We can either make a sketch or use one of the figures already constructed for the graphical solution (Figure 3.2). Starting from the given data we develop a formulation for the solution in as systematic a manner as possible; in this case a *top-down* formulation (Input 1). The given variables are V, S; the target variables are a, h; the auxiliary variables are B, LS, T and h_T. A set of eight equations is formulated in these eight variables. These are:

$V = 50$	Volume of pyramid (given)
$S = 100$	Surface area of pyramid (given)
$V = B\,h / 3$	Volume of pyramid as (1/3) x base x height
$S = B + LS$	Surface area of pyramid as area of base + area of four triangular sides
$B = a^2$	Base area for square pyramid of side length a
$LS = 4T$	Surface area of the four triangular sides (Lateral Surface)
$T = (a/2)\,h_T$	Surface area of one triangular side
$h_T^2 = (a/2)^2 + h^2$	Height of a triangular side (using Pythagoras‘ theorem)

There are four solutions of which only two correspond to positive solutions of a and h. We illustrate a variant auto-solver – **NSolvePositive** – which does not produce the two extraneous negative roots which would otherwise appear (Input and Output 1).

```
NSolvePositive[{V == 50, S == 100,
                V == 1/3 B h,
                S == B + LS,
                B == a^2,
                LS == 4 T,
                T == a/2 hT,
                hT^2 == (a/2)^2 + h^2},
               {a, h},
               {B, LS, T, hT}]

{{h == 3.92375, a == 6.18294}, {h == 12.7429, a == 3.43092}}
```

Input and Output 1

One of the two solutions is already known, having been found by graphical means (Figure 3.1). The other solution was overlooked while constructing the graphical solution. As well as the flat but broad pyramid there is a higher narrower one, both of which match the given data for surface and volume.

Algebraic solution:

To get a grasp of the class of all problems represented by the particular solution we need to determine the exact general solution for given S and V. From this we can obtain the conditions for solvability, that is, relations between S and V for which meaningful solutions exist; according to the requirements of the problem these must be real positive solutions.

We only need to remove the ‘N’ from **NSolveReal** (or **NPositiveReal**) and remove the two assignments $V = 50$, $S = 100$ (Input 2). The exact resolution of the general formulation leads to four solutions (Output 2), of which only the second and the fourth are acceptable. (This raises the question of why there are four solutions. The explanation is that manual solving the formulation for *a* leads to a quartic equation.)

$$\textbf{SolveReal}\Big[\Big\{V == \frac{1}{3} B\, h,\ S == B + LS,\ B == a^2,\ LS == 4\,T,\ T == \frac{a}{2} h_T,\ h_T^2 == \Big(\frac{a}{2}\Big)^2 + h^2\Big\},\ \{a, h\},\ \{B, LS, T, h_T\}\Big]$$

$$\Big\{\Big\{h == \frac{S^2 - \sqrt{S^4 - 288\, S\, V^2}}{24\, V},\ a == -\frac{\sqrt{S^2 + \sqrt{S^4 - 288\, S\, V^2}}}{2\sqrt{S}}\Big\},$$

$$\Big\{h == \frac{S^2 - \sqrt{S^4 - 288\, S\, V^2}}{24\, V},\ a == \frac{\sqrt{S^2 + \sqrt{S^4 - 288\, S\, V^2}}}{2\sqrt{S}}\Big\},$$

$$\Big\{h == \frac{S^2 + \sqrt{S^4 - 288\, S\, V^2}}{24\, V},\ a == -\frac{\sqrt{S^2 - \sqrt{S^4 - 288\, S\, V^2}}}{2\sqrt{S}}\Big\},$$

$$\Big\{h == \frac{S^2 + \sqrt{S^4 - 288\, S\, V^2}}{24\, V},\ a == \frac{\sqrt{S^2 - \sqrt{S^4 - 288\, S\, V^2}}}{2\sqrt{S}}\Big\}\Big\}$$

Input and Output 2

Note that in problems where we cannot easily recognise the acceptable solutions among the general ones, we only need to substitute an approximating command (e.g. **NSolveReal** for **SolveReal**) to the exact solutions and supply given data.

Condition for real solutions:

$$S^4 - 288 \times S \times V^2 \geq 0, \quad \text{or} \quad S^3 \geq 2 \times (12 \times V)^2$$

Conditions for positive real solutions:

$$S^2 - (S^4 - 288 \times S \times V^2)^{1/2} > 0 \text{ and } 288 \times S \times V^2 > 0$$

which are satisfied for $S > 0$ and $V > 0$.

Solution by the Simulation Method:

We only display the printout of the process of solution using DERIVE. Lines 20-23 show the defined calculation macros; the specific calculation task is solved at the end.

#1: $V = \frac{1}{3} \cdot a^2 \cdot h$

#2: $S = a^2 + 4 \cdot \frac{hT \cdot a}{2}$

#3: $hT^2 = \left(\frac{a}{2}\right)^2 + h^2$

#4: $\left[hT = \frac{\sqrt{(a^2 + 4 \cdot h^2)}}{2},\ hT = -\frac{\sqrt{(a^2 + 4 \cdot h^2)}}{2}\right]$

#5: $hT = \frac{\sqrt{(a^2 + 4 \cdot h^2)}}{2}$

#6: $S = a^2 + 4 \cdot \frac{\frac{\sqrt{(a^2 + 4 \cdot h^2)}}{2} \cdot a}{2}$

#7: $S = a \cdot \sqrt{(a^2 + 4 \cdot h^2)} + a^2$

#8: $\left[h = \frac{3 \cdot V}{a^2}\right]$

#9: $S = a \cdot \sqrt{\left(a^2 + 4 \cdot \left(\frac{3 \cdot V}{a^2}\right)^2\right)} + a^2$

#10: $S - a^2 = \frac{\sqrt{(36 \cdot V^2 + a^6)}}{a}$

#11: $a \cdot (S - a^2) = \sqrt{(36 \cdot V^2 + a^6)}$

#12: $(a \cdot (S - a^2) = \sqrt{(36 \cdot V^2 + a^6)})^2$

#13: $a^2 \cdot (S - a^2)^2 = 36 \cdot V^2 + a^6$

#14: $S^2 \cdot a^2 - 2 \cdot S \cdot a^4 + a^6 = 36 \cdot V^2 + a^6$

#15: $S^2 \cdot a^2 - 2 \cdot S \cdot a^4 = 36 \cdot V^2$

#16: $0 = -S^2 \cdot a^2 + 2 \cdot S \cdot a^4 + 36 \cdot V^2$

#17: $\left[a = \frac{\sqrt{(\sqrt{(S \cdot (S^3 - 288 \cdot V^2))} + S^2)}}{2 \cdot \sqrt{S}},\ a = -\frac{\sqrt{(\sqrt{(S \cdot (S^3 - 288 \cdot V^2))} + S^2)}}{2 \cdot \sqrt{S}},\right.$

$\left. a = \frac{\sqrt{(\sqrt{(S \cdot (S^3 - 288 \cdot V^2))} - S^2)}}{2 \cdot \sqrt{(-S)}},\ a = -\frac{\sqrt{(\sqrt{(S \cdot (S^3 - 288 \cdot V^2))} - S^2)}}{2 \cdot \sqrt{(-S)}}\right]$

$$\#18:\ h = \frac{S^2 - \sqrt{S \cdot (S^3 - 288 \cdot V^2)}}{24 \cdot V}$$

$$\#19:\ h = \frac{\sqrt{S \cdot (S^3 - 288 \cdot V^2)} + S^2}{24 \cdot V}$$

$$\#20:\ a1(S,\ V) := \frac{\sqrt{\sqrt{S \cdot (S^3 - 288 \cdot V^2)} + S^2}}{2 \cdot \sqrt{S}}$$

$$\#21:\ a2(S,\ V) := \frac{\sqrt{\sqrt{S \cdot (S^3 - 288 \cdot V^2)} - S^2}}{2 \cdot \sqrt{-S}}$$

$$\#22:\ h1(S,\ V) := \frac{S^2 - \sqrt{S \cdot (S^3 - 288 \cdot V^2)}}{24 \cdot V}$$

$$\#23:\ h2(S,\ V) := \frac{\sqrt{S \cdot (S^3 - 288 \cdot V^2)} + S^2}{24 \cdot V}$$

$\#24:\ a1(100,\ 50)$

$\#25:\ \sqrt{5 \cdot \sqrt{7} + 25}$

$\#26:\ 6.18294$

$\#30:\ h1(100,\ 50)$

$$\#31:\ \frac{25}{3} - \frac{5 \cdot \sqrt{7}}{3}$$

$\#32:\ 3.92374$

$\#27:\ a2(100,\ 50)$

$\#28:\ \sqrt{25 - 5 \cdot \sqrt{7}}$

$\#29:\ 3.43092$

$\#33:\ h2(100,\ 50)$

$$\#34:\ \frac{5 \cdot \sqrt{7}}{3} + \frac{25}{3}$$

$\#35:\ 12.7429$

Printout from Derive

Final remark:

The different methods complement each other, since various, but essential aspects of solving calculation problems are taken into account.

The solution methods should not be viewed in isolation; altogether they provide a comprehensive computer supported treatment of calculation tasks.

2.4 Some final comments

Comment 1: Of course KOERPERGEOMETRIE isn't suitable for the solution of all (open) spatial geometrical tasks. For example the discovery of all the convex polyhedra whose surface consists of equilateral triangles and squares could be successfully executed with suitable materials.

Comment 2: The use of adequate computer tools for the production of spatial geometrical objects opens up new possibilities for the solution of open problems.
The computer is an important medium

– for the extension of knowledge fields which can be treated only with great difficul ties in the lesson with conventional media
– for the reinforcements of creative intellectual performances
– for the flexible and economic reorganization of working processes and results.

Comment 3: What is the significance of computer represented spatial geometry for the student –for understanding our world, for private life and for future professional life? The spatial geometry lesson could for example make the connection to the numerous 3-D computer graphics applications being increasingly presented to the naive user in the form of aesthetic animations, 3-D games, 3-D CAD tools (for example for living space planning) and virtual-reality applications. In connection with that, the research question –still to be answered – arises: How can "spatial ability and imagination" be developed and exercised by the use of spatial geometry software? Of course the development and training of spatial ability and imagination is an original task of the spatial geometry lesson!

Comment 4: How might computer supported learning of spatial geometry look like in the near future?
Learning of spatial geometry thruogh virtual realities: The use of today's spatial geometry programs still has student and computer system left separated: The student can only indirectly communicate with the computer for example using a 'mouse'; therefore her/his kinesthetic feelings and experiences are very restricted; he/she executes options and watches the (spatial) result on the planar screen; the spatial interpretation can only be improved by stereographic representations and the use of red-

green spectacles.The computer generation of so-called virtual realities partially overcomes the limits between the student and computer system:
By means of a suitable audio-visual interface and the tactile interface data glove it is possible that the student has the (illusory) sensation to move and act with her/his whole body inside a simulated three dimensional world –for example carrying out virtual operations on objects of the virtual reality.
The following scenario for a future geometry learning is conceivable: The geometry learner proceeds as a Cybernaut in a three-dimensional geometrical world, for example in one for investigating polyhedra. She/he goes for a walk among the solids, looks at them from the worm’s-eye or the bird's-eye view, climbs on the solids around, feels the pointed corners and the sharp edges, she/he slips down the slippery solid faces, he penetrates the solids and views them from inside; she/he moves the solids, combines them, folding them down, changes their size, deforms them arbitrarily and carries out operations on them, for example section operations or she/he takes the role of a solid and for example 'experiences' the rolling as a solid etc.
How will such a computer represented spatial geometry impress the 'imagination' of the students? How will this change their relation to the real three-dimensional world surrounding?

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Doerfler, W. **The computer as a cognitive tool and cognition medium.** Der Computer als kognitives Werkzeug und kognitives Medium. Doerfler, W. et al. (eds.) (1991) Computer - Mensch - Mathematik . Stuttgart: B.G. Teubner

Kaput, J.J. **Technology an Mathematics Education.** Douglas, A.G. (ed.) (1992) Handbook of Research on Mathematics Education. New York: Simon & Schuster

King, J.R.; Schattschneider, D. (eds.) (1997): **Geometry Turned On! Dynamic Software in Learning, Teaching and Research.** Washington, DC: The Mathematical Association of America

Villiers, M. de (1997): **The Future of Secondary School Geometry**.
Pythagoras, v. 44, Dec. 1997, p 37-5

3.3 Cabri-géomètre: a selection of publications

3.3.1 Books

French

Expérimenter en mathématiques avec Cabri-géomètre : Utilisation en lycée
Tome 1, Activités pour classes de 2° et 1°, 104 fiches A4
Tome 2, Complément pour professeur, 128 p, disquette Mac ou PC
MARTIN, Y. (1994) Editions Archimède, Argenteuil.

Cabri-classe, apprendre la géométrie avec un logiciel (Collège)
275 p A4, disquette Mac ou PC
CAPPONI, B. ;LABORDE C. (1994), Éditions Archimède, Argenteuil.

Cabricolages : exploration dans le monde de la géométrie plane
Livre de l'élève, 68 p
Livre du maitre, commentaires didactiques, 144 p, disquette
CHASTELLAIN, M.; LUGON, S. (1992). Editions LEP (Loisir et Pédagogie S.A.), CH-1052, Le Mont sur Lausanne, Suisse. Tel. (++4121/021) 653.53.30 - Fax (++4121/021) 653.57.51

Faire de la géométrie en jouant avec Cabri-géomètre
Tome 1, pp 1-208
Tome 2, pp 209-480 p
CUPPENS, R. (1996), Editions Archimède, Argenteuil.

Ces livres sont les brochures N°104 et 105 de l'APMEP.
APMEP, 26 rue Duméril 75013 Paris (http://www.univ-lyon1.fr/apmep/).

Faire de la géométrie supérieure en jouant avec Cabri-géomètre II
Tome 1, pp 1-168
Tome 2, pp 169-277
CUPPENS, R. (1999), Editions Archimède, Argenteuil.
Ces livres sont les brochures N°124 et 125 de l'APMEP.
APMEP, 26 rue Duméril 75013 Paris (http://www.univ-lyon1.fr/apmep/).
Disponible également au laboratoire Leibniz, Grenoble.

Introduction à la géométrie avec la TI-92
DAHAN, J.J. (1998) 240 p. Ellipses-Edition Marketing, 32 rue Bargues, 75015 Paris.
Disponible au laboratoire Leibniz, Grenoble.

Dessiner l'espace ou Comment employer Cabri-géomètre en géométrie dans l'espace
ROUSSELET, M. (1995),126 p. Editions Archimède, Argenteuil.

Apprivoiser la géométrie avec Cabri-géométre

CHARRIERE, P-M. (1996), 234 p, disquette, Monographie du Centre Informatique Pédagogique de Genève N° 4, C.I.P., case postale 3144, CH-1211 Genève 3, Suisse.
Available from Laboratoire Leibniz, Grenoble.

CINÉ-MAthématiques avec Cabri-Géomètre
BENEDETTI, C. (1999) Edition : Ministère de la Communauté Française, Centre Technique et Pédagogique de l'enseignement de la Communauté française.

Actes de l'Université d'été : Apprentissage et enseignement de la géométrie avec ordinateur : Utilisation du logiciel Cabri-géomètre en classe
CAPPONI, B. ; LABORDE, C. (eds) (1994) 170 p. IREM de Grenoble, domaine universitaire, 38402 Saint Martin d'Hères (http://www.ac-grenoble.fr/irem/)
Available from Laboratoire Leibniz, Grenoble.

Actes de l'Université d'été : Cabri-géomètre, de l'ordinateur à la calculatrice. De nouveaux outils pour l'enseignement de la géométrie
CAPPONI, B. ; LABORDE, J.M. (eds) (1998) 256 p. Grenoble : IREM et IUFM de Grenoble.
Available from Laboratoire Leibniz, Grenoble.

Figures d'optique animées par Cabri-géomètre
ARRAGON, M. (1995). CNDP, Paris (http://www.cndp.fr/) ou CRDP de Grenoble, 11, Av. Ge Champon, 38000 Grenoble (http://www.ac-grenoble.fr/crdp/).

German

Schulgeometrisches Konstruieren mit dem Computer
SCHUMANN, H. (1991), Metzler u. Teubner.

Cabricolages, Expeditionen in die Welt der ebenen Geometrie
CHASTELLAIN, M. ; LUGON, S. ; ATZBACH, R. (1993) CoMet, Duisburg, Allemagne.

Arbeitsbuch CABRI Géomètre, Kontruieren mit dem Computer
HENN, H. W. ; JOCK, W. (1993) DÜMMLER, Bonn, Allemagne.

Cabri-Géomètre, eine Evaluation durch Lehrer
SCHUMANN H. (1994), Cornelsen Software.

Im Zugmodus der Cabri-Geometrie.
Interaktionsstudien zum Mathematiklernen mit dem Computer
HOELZL, R. (1994), Weinheim: Deutscher Studien-Verl.

Geometrie beweglich mit Cabri Géomètre II
ELSCHENBROICH, H.-J. ; Noll,G. (2000), Dümmler

English

The two first books are parts of the US package for Cabri Geometry II.
Activities for Cabri Geometry II

MYERS, D.L.(1994), 42 p, developped by Houghton Mifflin Company, Texas Instruments Inc, Dallas, US.

Explorations for the Mathematics Classroom using Cabri Geometry II
VONDER EMBSE, C., ENGEBRETSEN, A. (1994), 60 p, Texas Instruments Inc, Dallas, US.

92 geometric explorations on the TI 92
KEYTON, M.(1996) 184 p. Texas Instruments Inc, Dallas, US.

Exploring the basics of geometry with Cabri
WILGUS, W., PIZZUTO, L. (1997) 102 p, Texas Instruments Inc., Dallas, US.

Geometric Investigations for the Classroom using the TI 92
VONDER EMBSE, C., ENGEBRETSEN, A.(1996), 60 p, Texas Instruments Inc, Dallas, US.

Geometrical investigations : a Companion to Cabri-Géomètre (book and/or on disc for Macintosh)
CLARK, G., REDDEN, E. (AAMT Australian Association of Mathematics Teachers), Australia. (http://www.aamt.edu.au)

Geometrical investigations : a Companion to Cabri II (book and/or on disc for Macintosh or PC Windows)
CLARK, G. ; REDDEN, E., AAMT, Australie.

Geometry and Trigonometry with Cabri-Géomètre (with disc for PC or Macintosh)
FOSTER, P. ; SAUNDERS, R., AAMT, Australie.

Geometry with Cabri : Exploring Trigonometry
Little, C. ; Sutherland, R. (1995) 28 p. photocopiable ressource. Chartwell-Yorke, 114 High Street, Belmont, Bolton, Lancashire, BL7 8AL, England, UK.
(http://www.chartwellyorke.com/)

Geometry with Cabri : Taking A New Angle
Little, C. ; Sutherland, R. (1995) 23 p. photocopiable ressource. Chartwell-Yorke, UK.

Geometry with Cabri : Transforming, Transformations
Little C., Sutherland R. (1995) 36 p. photocopiable ressource. Chartwell-Yorke, UK.

Discovering Geometry with a computer - using Cabri Géomètre
SCHUMANN, H., GREEN D. (1994) Chartwell-Yorke, UK.

Some Adventures in Euclidean Geometry
De VILLIERS, M.(1994,1996) 214 p., University of Durban-Westville, South Africa.

Intelligent Learning Environments, the case of geometry
LABORDE, J.-M. (ed.) (1994) NATO ESI Series, Springer Verlag.

Cabri geometry II, Geometry for the world
LABORDE, C. ; KEYTON, M. ; HOELZL ; R., KOBAYASHI, I ; LABORDE, J.M. ; HASSAL, M. ; GEIGER, V. ; TURNAU, S. ; VONDER EMBSE, C. ; ENGEBRETSEN, A. (1996) 36 p, Texas Instruments Inc, Dallas, US.

Geometric explorations for the classroom
ENGEBRETSEN, A. ; Jahr, C., LABORDE, J-M. ; OLMSTEAD G. ; VONDER EMBSE, C. (1997) 38p., NCTM National Conference, Minneapolis, Minnesota, April 17, 1997.

The T3 program (Teachers Teaching with Technology, WAITS B., DEMANA F., 1986) has some documents related to Cabri Geometry II, for example :

Analytic Geometry Institute – Summer 1998
VONDER EMBSE, C. ; OLMSTEAD, G. ; GARRISON, G. ;WORTMAN, J. ; KEYTON, M. ; ENGEBRETSEN, A. ; HICKS, J. (1998) 176 p,   office)

Spanish

Explorando la geometria en los CLUBES CABRI
BONOMO, F. ; D'ANDREA, C. ; LAPLAGNE, S. ; SZEW, M.(1996) 140 p., Red Olimpica, Buenos Aires, Argentina.

Matemáticas con Cabri II
MORA SANCHEZ ; J. A. (1999) 56 p., Proyecto Sur de Ediciones, S. L., Armilla (Granada), España.

Curso de Geometría para el bachillerato con un acercamiento informal usando Cabri-Géomètre
CORTÉS ZAVALA, J.C. ; LOPEZ ZAMUDIO, A. (1999) 160 p., Universidad Michoacana de San Nicolás de Hidalgo, Michoacán, México

Italian

Invito alla geometria con Cabri-géomètre
PELLEGRINO, C. ; ZAGABRIO, M-G. (1996) 138 p. Proposta di lavaro per la scuola secondaria superiore, Collana strumenti didattici, Iprase del Trentino, Via S. Margherita, Trento, Italia.

Portuguese

Explorando Geometria elementar com o dinamismo do Cabri-Géomètre,
SANGIACOMO ; L., DA SILVA, M.C.L. ; DE OLIVEIRA, M.C.A. ; DE SOUZA, V.H.G. (1999) 109 p. PROEM Editoria Ltda, São Paulo.

Explorando os polígonos nas séries iniciais do ensino fundamental (versão preliminar)
MAGINA, S. ; DA COSTA, N.L. ; HEALY, L. ; PIETROPAOLO, R. (1999) 87 p. PROEM Editoria Ltda, São Paulo.

Geometria com Cabri-géomètre : diferentes metodologias
SANGIACOMO, L. ; DE OLIVEIRA, M.C.A. ; MIGUEL, M.I.R. ; DE SOUZA, V.H.G. (1999) 29 p. PROEM Editoria Ltda, São Paulo.

Descobrindo o Cabri-géomètre, Caderno de actividades
BONGIOVANNI, V. ; CAMPOS, T. ; ALMOULOUD, S. (1998) São Paulo : Editoria FTD S.A.

Explorando conceitos de geometria elementar com o software Cabri-Géomètre
DA SILVA, M.C.L. ; ALMOULOUD, S. ; CAMPOS, T. ; BONGIOVANNI, V. (1998)
84 p, disquete. EDUC, Rua Monte Alegre, 984, Perdizes, Sao Paulo, SP.

3.3.2 Communications at congresses etc. (a selection on English)

Cabri-géomètre vs. The Geometer's Sketchpad : A Comparison of Two Dynamic Geometry Systems
HABEGGER, W. V ; EMERT, J.W. (1993) Notices of the American Mathematical Society 40 (8) pp. 988-992, Providence, Rhode Island, USA.

Exploring Non-Euclidean Geometry in a Dynamic Environment like Cabri-géomètre
Laborde J-M. (1997) In: King, J. & Schattschneider, D. (eds) Geometry Turned On ! Dynamic Software in Learning, Teaching, and Research (Chap IV) (pp. 185-191). Providence, USA : MAA Publications.

Producing and Using Loci with Dynamic Geometry Software
SCHUMANN, H. ; GREEN, D. (1997) In: King J. & Schattschneider D. (eds) Geometry Turned On ! Dynamic Software in Learning, Teaching, and Research (Chap II) (pp. 79-87). Providence, USA : MAA Publications.

Using Geometry to Model and Explore Functions
Bellemain, F. ; CAPPONI, B.; GREMILLION, D. (1995)
Eightysomething! Vol. 5 (1), 8-9, Texas Instruments.

Geometrical Tools
MASON, J. (1992) Micro Math **8** (3), Association of Teachers of Mathematics.

Pythagoras in Cabri-géomètre
MASON ; J. et Q. (1991) Micro MATH, vol.7, n°2, 15-17.

Dossier Cabri-géomètre dans **Micro MATH, vol.8, n°2**

What is Cabri-géomètre?
BLOOMFIELD A. (1992) Micro MATH, vol.8, n°2, 23-24.

Approaches to Cabri
ROGERS L. (1992) Micro MATH, vol.8, n°2, 25-26.

Cabri-géomètre and nomograms
FAUX G. (1992) Micro MATH, vol.8, n°2, 27-29.

Trigonometry with Cabri in a French Classroom
Capponi, B. ; SUTHERLAND, R. (1992) Micro MATH, vol.8, n°2, 32-33.

Initial reactions
PHILLIPS, R., SELINGER, M. (1992) Micro MATH, vol.8, n°2, 34-36.

Curricular Configurations
TAHTA, D. (1992) Micro MATH, vol.8, n°2, 37-39.

3.3.3 Journals dedicated to Cabri-géomètre

French

Cabriole, le journal des utilisateurs de Cabri-géomètre
(1992) n°1-Juin / n°2-Novembre.
(1993) n°3-Février / n°4-Mai / n°5-Déc.
(1994) n°6-Mai.
(1995) n°7-Février / n°8- Oct. / n°9-Nov.

Abracadabri, Osez la géométrie avec Cabri
(1993) Novembre-Décembre, n°0.
(1994) Janvier-Février, n°1 / Mars-Avril, n°2 / Mai-Juin, n°3 / Juillet-Août, n°4 / Septembre-Octobre, n°5 / Novembre-Décembre, n°6.
(1995) Janvier-Février, n°7 / Mars-Avril, n°8 / Mai-Juin, n°9 / Juillet-Août , n°10 / Septembre-Octobre, n°11.

The following journals are distributed only in the Web:

Cabriole : http://www-cabri.imag.fr/cabriole/

Abracadabri : http://www-cabri.imag.fr/abracadabri/
Anciens n° : http://www.ac-reunion.fr/pedagogie/covincep/icosaweb/GeomJava/Intro.htm

Envol du GRMS
(Revue du Groupe des Responsables en Mathématique au Secondaire) - Québec
La revue comporte une rubrique sur Cabri-Géomètre, par Gérald Saint-Amand (voir site La Cabri-Thèque au Québec : http://pages.infinit.net/cabri)
Les cinq premiers articles mentionnés sont consultables sur le site de la revue : http://cyberscol.qc.ca/partenaires/grms/envol/envol.html
Envol no. 94, "Du coq à l'âne ou du triangle rectangle à l'équation de la droite", février 1996,
no. 95, "Un jeu d'enfants : de la roue au SpiroGraph", avril 1996,
no. 96, "Cabri au service du hockey", juin 1996,
no. 97, "Cabri vs TI-92", novembre 1996,
no. 98 , "Boîtes noires", février 1997,
no. 99 , "Géométrie logique"
no. 100, "A la rencontre des vecteurs"

Italian

CABRIRRSAE, Bolettino degli utilizzatori di Cabri-géomètre
(1993) Nov.-n°0 / Février-n°1.
(1994) Mai-n°2 / Oct.- n°3.
(1995) Février-n°4 / Mai-n°5 / Octobre n°6.
(1996) Février-n°7, Mai- n°8, Sept.-n°9, Déc.-n°10.
(1997) Juin-n°12 (Numéro spécial), Septembre-n°13 (2° Numéro spécial), Décembre-n°14.
(1998) Mars-n°15, Juin-n° 16.

Quaderni di CABRIRRSA
Cabri-géomètre e il foglio elettronico, Août 1994, N°3.
Cabri-géomètre e le trasformazioni geometriche, Octobre 1994, N°4.
Algebra con Cabri-géomètre, Novembre 1994, N°5.
La sezione aurea di un segmento e applicazioni, Novembre 1994, N°6.
La misura in Cabri, Avril 1995, N°7.
Cabri-géomètre e i luoghi geometrici, Octobre 1995, N°8.
Sezioni piane di un cubo : un problema di geometria dello spazio risolto con Cabri-géomètre, N°9.
Cabri come strumento di esplorazione della geometria non euclidea, N°10.
Reviews of these articles: http://arci01.bo.cnr.it/cabri/

3.3.4 INTERNET: Addresses and sites
Laboratoire Leibniz, 46 av. Félix Viallet, 38000 Grenoble, France
Le site Web de **Cabri** est :http://www-cabri.imag.fr ou http://www.cabri.net
Editions Archimède, 5, rue Jean Grandel, 95100 Argenteuil
IREM de Grenoble , Domaine universitaire, 38402 Saint Martin d'Hères :
http://www.ac-grenoble.fr/irem/
abraCAdaBRI : http://www-cabri.imag.fr/abracadabri/
Cabriole : http://www-cabri.imag.fr/cabriole/
La Cabri-Thèque au Québec : http://pages.infinit.net/cabri
Cabri au **collège Jules Flandrin** de Corenc (France) :
http://www-cabri.imag.fr/TeleCabri/PassionRecherche/
Cabri au **collège Vincendo** à la Réunion (France)
http://www.ac-reunion.fr/pedagogie/covincep/Frames/F_Cabri/M_Pedago.htm
Cabri Geometry du **Math Forum** à **Swarthmore** (Etats-Unis)
http://forum.swarthmore.edu/cabri/cabri.html
Cabri-Géomètre depuis **Bologne** (Italie) : http://arci01.bo.cnr.it/cabri/
Cabri Geometry II : le site officiel de **Texas Instruments** à Dallas (Etats-Unis)
http://www.ti.com/calc/docs/cabri.htm
Un nouveau site en français sur l'utilisation de la TI-92 en cours :
http://www-cabri.imag.fr/nathalie/ti92/ti92.htm
Optique et Cabri de **l'Université de Nantes** (France)
http://www.sciences.univ-nantes.fr/physique/enseignement/tp/optique/index.html
Physique et Cabri à **l'Université de Provence**, Marseille (France)
http://www.up.univ-mrs.fr/~laugierj/indexc.html
IcosaWEB serveur mathématique de La Réunion (France)
http://www.ac-reunion.fr/pedagogie/covincep/icosaweb/HomeJS.htm
Archives Cabri de **CIGS** (Corner for Interactive Geometry Software)
http://forum.swarthmore.edu/cabri/cabri.html
Cabri-géomètre au Brésil : http://www.cabri.com.br
Cabri-géomètre en Hollande : http://www.pandd.demon.nl/cabri.htm

More sites using Cabri:
Coniques de Stothers : http://www.maths.gla.ac.uk/~wilson/cabripages/cabri0.html
Géométrie Hyperbolique de Tim Lister : http://mcs.open.ac.uk/tcl2/nonE/nonE.html
Famille de Coniques de Lee Dickey :
http://forum.swarthmore.edu/dynamic/submissions/familyofconics/index.html
Machines mathématiques de l'Université de Modène (Italie) :
http://www.museo.unimo.it/theatrum/
Le site de **Pierre Crespin** du Lycée Dumont d'Urville à Toulon (France) :
http://picre.citeweb.net/

Le site de **José Antonio Mora** (Alicante, Espagne) : mécanismes, coordonnées
http://teleline.terra.es/personal/joseantm
Le site d'un **professeur japonais** (Japon) :
http://www2.gunmanet.or.jp/mow/math/cabri/index_e.htm

3.4 The Geometer's Sketchpad: a selection of publications

3.4.1 Books

Sanders, C. (1994) : **Perspective Drawing with The Geometer's Sketchpad.** Berkeley, CA: Key Curriculum

Shaffer, D. (1995) : **Exploring Trigonometry with The Geometer's Sketchpad.** Berkeley, CA: Key Curriculum

Bennett, D. (1995) : **Pythagoras Plugged In. Proofs and Problems for The Geometer's Sketchpad.** Emeryville, CA: Key Curriculum

Scher, D. (1995) : **Exploring Conic Sections with The Geometer's Sketchpad.** Berkeley, CA: Key Curriculum

King, J.R. (1996) : **Geometry Through the Circle with The Geometer's Sketchpad.** Berkeley, CA: Key Curriculum

Battista, M.T. (1998) : **Shape Makers: Developing Geometric Reasoning with The Geometer's Sketchpad.** Emeryville, CA: Key Curriculum

Wyatt, K.W. et al. (1998) : **Geometry activities for Middle School Students with The Geometer's Sketchpad.** Berkeley, CA: Key Curricu lum

Villiers, M.D. de (1999) : **Rethinking Proof with The Geometer's Sketchpad.** Emeryville, CA: Key Curriculum

Bennett, D. (1999) : **Exploring Geometry with The Geometer's Sketchpad.** Emeryville, CA: Key Curriculum

3.4.2 Selected main internet adresses and sites:

http://www.keypress.com/sketchpad
http://www.mathforum.com

3.5 Cinderella: a selection of publications

Richter-Gebert, J.; Kortenkamp, U.H. (1999):
The Interactive Geometry Software Cinderella

Richter-Gebert, J.; Kortenkamp, U.H. (2000):
User Manual for the Interactive Geometry Software Cinderella

Richter-Gebert, J.; Kortenkamp, U.H. (2000):
Die Interaktive Geometrie-Software Cinderella

Mathe Geometrie 5.-10. Klasse. Stuttgart: Klett 2000 (with CD ROM)
Internet addresses:
http://www.cinderella.de
http://www.maa.org/reviews/cinderella.html

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